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RESEARCH ARTICLE

ON THE ZEROS OF THE POLAR DERIVATIVE OF A POLYNOMIAL

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ABSTRACT

In this paper we find bounds for the zeros of the polar derivative of a polynomial under certain conditions on the coefficients. Our results generalize many known results in this direction.

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INTRODUCTION

For any polynomial P(z) of degree n, the polar derivative of P(z) with respect to a positive real number α is defined by $D_\alpha P(z) = nP(z) + (\alpha - z)P'(z)$ which is a polynomial of degree at most n-1. The polar derivative generalizes the ordinary derivative in the sense that $\lim_{\alpha\to\infty} \frac{D_\alpha P(z)}{\alpha} = P'(z)$. In the context of the famous Enestrom-Kakeya Theorem (Marden, 1966) which states that all the zeros of the polynomial $P(z) = \sum_{j=0}^{n} a_j z^j$ with $a_n \ge a_{n-1} \ge ...... \ge a_1 \ge a_0 > 0$ lie in $|z| \le 1$, attempts have been made to find bounds for the zeros of $D_\alpha P(z)$ under certain conditions on its coefficients. In this connection recently Ramulu *et al.,* (2015) proved the following results:

Theorem A: Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n and $D_\alpha P(z) = nP(z) + (\alpha - z)P'(z)$ be a polar derivative of P(z) with respect to a real number α such that
$na_0 \le (n-1)a_1 \le (n-2)a_2 \le ... \le 3a_{n-1} \le 2a_{n-2} \le a_{n-1}$.
If $\alpha = 0$, then all the zeros of the polar derivative $D_0 P(z)$ lie in $|z| \le \frac{1}{|a_{n-1}|}[a_{n-1} - na_0 + |na_0|]$.

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Theorem B: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n and $D_\alpha P(z)=nP(z)+(\alpha-z)P'(z)$ be a polar derivative of P(z) with respect to a real number α such that

$na_0 \geq (n-1)a_1 \geq (n-2)a_2 \geq ... \geq 3a_{n-1} \geq 2a_{n-2} \geq a_{n-1}$.

If $\alpha=0$, then all the zeros of the polar derivative $D_0 P(z)$ lie in $|z| \leq \frac{1}{|a_{n-1}|}[|na_0| + na_0 - a_{n-1}]$.

Later Reddy *et al.* (2015) proved the following generalizations of Theorems A and B:

Theorem C: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n and $D_\alpha P(z)=nP(z)+(\alpha-z)P'(z)$ be a polar derivative of P(z) with respect to a real number α such that

$(i+2)\alpha a_{i+2} + \{n-(i+1)\}\alpha a_{i+1} \geq (i+1)\alpha a_{i+1} + (n-i)a_i, i=0,1,2,......,n-2$.

Then all the zeros of the polar derivative $D_\alpha P(z)$ lie in

$$|z| \leq \frac{1}{|n\alpha a_n + a_{n-1}|}[n\alpha a_n + a_{n-1} - (\alpha a_1 + na_0) + |\alpha a_1 + na_0|].$$

Theorem D: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n and $D_\alpha P(z)=nP(z)+(\alpha-z)P'(z)$ be a polar derivative of P(z) with respect to a real number α such that

$(i+2)\alpha a_{i+2} + \{n-(i+1)\}\alpha a_{i+1} \leq (i+1)\alpha a_{i+1} + (n-i)a_i, i=0,1,2,......,n-2$.

Then all the zeros of the polar derivative $D_\alpha P(z)$ lie in

$$|z| \leq \frac{1}{|n\alpha a_n + a_{n-1}|}[|\alpha a_1 + na_0| + (\alpha a_1 + na_0) - n\alpha a_n - a_{n-1}].$$

In this paper we find lower bounds for the zeros of $D_\alpha P(z)$ under the same conditions. In fact, we prove the following results:

Theorem 1: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n and $D_\alpha P(z)=nP(z)+(\alpha-z)P'(z)$ be a polar derivative of P(z) with respect to a real number α such that

$(i+2)\alpha a_{i+2} + \{n-(i+1)\}\alpha a_{i+1} \geq (i+1)\alpha a_{i+1} + (n-i)a_i, i=0,1,2,......,n-2$.

Then all the zeros of the polar derivative $D_\alpha P(z)$ lie in $|z| \geq \frac{|\alpha a_1 + na_0|}{M}$, where

$M = |n\alpha a_n + a_{n-1}| + (n\alpha a_n + a_{n-1}) - (\alpha a_1 + na_0)$

Combining Theorem C and Theorem 1, we get the following result:

Theorem 2: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n and $D_\alpha P(z)=nP(z)+(\alpha-z)P'(z)$ be a polar derivative of P(z) with respect to a real number α such that

$(i+2)\alpha a_{i+2} + \{n-(i+1)\}\alpha a_{i+1} \geq (i+1)\alpha a_{i+1} + (n-i)a_i, i=0,1,2,......,n-2$.

Then all the zeros of the polar derivative $D_\alpha P(z)$ lie in

$$\frac{|\alpha a_1 + na_0|}{M} \leq |z| \leq \frac{1}{|n\alpha a_n + a_{n-1}|}[n\alpha a_n + a_{n-1} - (\alpha a_1 + na_0) + |\alpha a_1 + na_0|],$$

where M is as given in Theorem 1.

Taking $\alpha, a_i > 0, i=0,1,2,......,n$ in Theorem 2, we get the following result:

Corollary 1: Let $P(z)=\sum_{j=0}^{n}a_j z^j$ be a polynomial of degree n with positive coefficients and

$D_\alpha P(z)=nP(z)+(\alpha-z)P'(z)$ be a polar derivative of P(z) with respect to a real number α such that

$(i+2)\alpha a_{i+2}+\{n-(i+1)\}\alpha_{i+1}\geq(i+1)\alpha a_{i+1}+(n-i)a_i, i=0,1,2,......,n-2$.

Then all the zeros of the polar derivative $D_\alpha P(z)$ lie in $\frac{\alpha a_1+na_0}{2(n\alpha a_n+a_{n-1})-(\alpha a_1+na_0)}\leq|z|\leq 1$.

Theorem 3: Let $P(z)=\sum_{j=0}^{n}a_j z^j$ be a polynomial of degree n and $D_\alpha P(z)=nP(z)+(\alpha-z)P'(z)$ be a polar derivative of P(z) with respect to a real number α such that

$(i+2)\alpha a_{i+2}+\{n-(i+1)\}\alpha a_{i+1}\leq(i+1)\alpha a_{i+1}+(n-i)a_i, i=0,1,2,......,n-2$.

Then all the zeros of the polar derivative $D_\alpha P(z)$ lie in $|z|\geq\frac{|\alpha a_1+na_0|}{M'}$, where

$M'=|n\alpha a_n+a_{n-1}|-n\alpha a_n-a_{n-1}+\alpha a_1+na_0$.

Combining Theorem D and Theorem 3, we get the following result:

Theorem 4: Let $P(z)=\sum_{j=0}^{n}a_j z^j$ be a polynomial of degree n and $D_\alpha P(z)=nP(z)+(\alpha-z)P'(z)$ be a polar derivative of P(z) with respect to a real number α such that

$(i+2)\alpha a_{i+2}+\{n-(i+1)\}\alpha a_{i+1}\leq(i+1)\alpha a_{i+1}+(n-i)a_i, i=0,1,2,......,n-2$.

Then all the zeros of the polar derivative $D_\alpha P(z)$ lie in

$$\frac{|\alpha a_1+na_0|}{M'}\leq|z|\leq\frac{1}{|n\alpha a_n+a_{n-1}|}[|\alpha a_1+na_0|+(\alpha a_1+na_0)-n\alpha a_n-a_{n-1}],$$

where M' is as given in Theorem 3

Taking $\alpha, a_i>0, i=0,1,2,......,n$ in Theorem 3, we get the following result:

Corollary 2: Let $P(z)=\sum_{j=0}^{n}a_j z^j$ be a polynomial of degree n with positive coefficients and

$D_\alpha P(z)=nP(z)+(\alpha-z)P'(z)$ be a polar derivative of P(z) with respect to a real number α such that

$0<(i+2)\alpha a_{i+2}+\{n-(i+1)\}\alpha_{i+1}\leq(i+1)\alpha a_{i+1}+(n-i)a_i, i=0,1,2,......,n-2$.

Then all the zeros of the polar derivative $D_\alpha P(z)$ lie in

$$1\leq|z|\leq\frac{1}{n\alpha a_n+a_{n-1}}[2(\alpha a_1+na_0)-n\alpha a_n-a_{n-1}].$$

Proofs of Theorems

Proof of Theorem 1: We have

$$P(z)=a_n z^n+a_{n-1}z^{n-1}+......+a_2 z^2+a_1 z+a_0$$
$$D_\alpha P(z)=nP(z)+(\alpha-z)P'(z)$$
$$=(n\alpha a_n+a_{n-1})z^{n-1}+\{(n-1)\alpha a_{n-1}+2a_{n-2}\}z^{n-2}+\{(n-2)\alpha a_{n-2}+3a_{n-3}\}z^{n-3}$$
$$+......+\{3\alpha a_3+(n-2)a_2\}z^2+\{2\alpha a_2+(n-1)a_1\}z+(\alpha a_1+na_0).$$

Now , consider the polynomial

$$F(z)=(1-z)D_\alpha P(z)$$

$$= -(n\alpha a_n + a_{n-1})z^n + \{n\alpha a_n + (1+\alpha - n\alpha)a_{n-1} - 2a_{n-2}\}z^{n-1} + \{(n-1)\alpha a_{n-1}$$
$$+ (2+2\alpha - n\alpha)a_{n-2} - 3a_{n-3}\}z^{n-2} + ... + \{3\alpha a_3 + (n-2-2\alpha)a_2 - (n-1)a_1\}z^2$$
$$+ \{2\alpha a_2 + (n-1-\alpha)a_1 - na_0\}z + (\alpha a_1 + na_0)$$
$$= (\alpha a_1 + na_0) + G(z),$$

where

$$G(z) = -(n\alpha a_n + a_{n-1})z^n + \{n\alpha a_n + (1+\alpha - n\alpha)a_{n-1} - 2a_{n-2}\}z^{n-1} + \{(n-1)\alpha a_{n-1}$$
$$+ (2+2\alpha - n\alpha)a_{n-2} - 3a_{n-3}\}z^{n-2} + ... + \{3\alpha a_3 + (n-2-2\alpha)a_2 - (n-1)a_1\}z^2$$
$$+ \{2\alpha a_2 + (n-1-\alpha)a_1 - na_0\}z$$

For $|z| \le 1$, we have by using the hypothesis,

$$|G(z)| \le |n\alpha a_n + a_{n-1}| + |n\alpha a_n + (1+\alpha - n\alpha)a_{n-1} - 2a_{n-2}| + |(n-1)\alpha a_{n-1} + (2+2\alpha$$
$$- n\alpha)a_{n-2} - 3a_{n-3}\ | + ... + |3\alpha a_3 + (n-2-2\alpha)a_2 - (n-1)a_1|$$
$$+ |2\alpha a_2 + (n-1-\alpha)a_1 - na_0|$$
$$= |n\alpha a_n + a_{n-1}| + n\alpha a_n + (1+\alpha - n\alpha)a_{n-1} - 2a_{n-2}$$
$$+ (n-1)\alpha a_{n-1} + (2+2\alpha - n\alpha)a_{n-2} - 3a_{n-3} + ...$$
$$+ 3\alpha a_3 + (n-2-2\alpha)a_2 - (n-1)a_1 + 2\alpha a_2 + (n-1-\alpha)a_1 - na_0$$
$$= |n\alpha a_n + a_{n-1}| + (n\alpha a_n + a_{n-1}) - (\alpha a_1 + na_0)$$
$$= M$$

Since G(z) is analytic for $|z| \le 1$ and G(0)=0, it follows by Schwarz Lemma that $|G(z)| \le M|z|$ for $|z| \le 1$.

Hence, for $|z| \le 1$,

$$|F(z)| = |(\alpha a_1 + na_0) + G(z)|$$
$$\ge |\alpha a_1 + na_0| - |G(z)|$$
$$\ge |\alpha a_1 + na_0| - M|z|$$
$$> 0$$

if

$$|z| < \frac{|\alpha a_1 + na_0|}{M}.$$

This shows that all the zeros of F(z) lie in $|z| \ge \dfrac{|\alpha a_1 + na_0|}{M}$.

Since the zeros of $D_\alpha P(z)$ are also the zeros of F(z), it follows that all the zeros of $D_\alpha P(z)$ lie in $|z| \ge \dfrac{|\alpha a_1 + na_0|}{M}$ and the proof of Theorem 1 is complete.

Proof of Theorem 3: We have

$$P(z) = a_n z^n + a_{n-1}z^{n-1} + ... + a_2 z^2 + a_1 z + a_0$$
$$D_\alpha P(z) = nP(z) + (\alpha - z)P'(z)$$
$$= (n\alpha a_n + a_{n-1})z^{n-1} + \{(n-1)\alpha a_{n-1} + 2a_{n-2}\}z^{n-2} + \{(n-2)\alpha a_{n-2} + 3a_{n-3}\}z^{n-3}$$

$+ ...... + \{3\alpha a_3 + (n-2)a_2\}z^2 + \{2\alpha a_2 + (n-1)a_1\}z + (\alpha a_1 + na_0)$.

Now, consider the polynomial

$$\begin{aligned}
F(z) &= (1-z)D_\alpha P(z) \\
&= -(n\alpha a_n + a_{n-1})z^n + \{n\alpha a_n + (1+\alpha - n\alpha)a_{n-1} - 2a_{n-2}\}z^{n-1} + \{(n-1)\alpha a_{n-1} \\
&\quad + (2+2\alpha - n\alpha)a_{n-2} - 3a_{n-3}\}z^{n-2} + ... + \{3\alpha a_3 + (n-2-2\alpha)a_2 - (n-1)a_1\}z^2 \\
&\quad + \{2\alpha a_2 + (n-1-\alpha)a_1 - na_0\}z + (\alpha a_1 + na_0) \\
&= (\alpha a_1 + na_0) + G(z),
\end{aligned}$$

where

$$\begin{aligned}
G(z) &= -(n\alpha a_n + a_{n-1})z^n + \{n\alpha a_n + (1+\alpha - n\alpha)a_{n-1} - 2a_{n-2}\}z^{n-1} + \{(n-1)\alpha a_{n-1} \\
&\quad + (2+2\alpha - n\alpha)a_{n-2} - 3a_{n-3}\}z^{n-2} + ... + \{3\alpha a_3 + (n-2-2\alpha)a_2 - (n-1)a_1\}z^2 \\
&\quad + \{2\alpha a_2 + (n-1-\alpha)a_1 - na_0\}z
\end{aligned}$$

For $|z| \le 1$, we have by using the hypothesis,

$$\begin{aligned}
|G(z)| &\le |n\alpha a_n + a_{n-1}| + |n\alpha a_n + (1+\alpha - n\alpha)a_{n-1} - 2a_{n-2}| + |(n-1)\alpha a_{n-1} + (2+2\alpha \\
&\quad - n\alpha)a_{n-2} - 3a_{n-3}| + ... + |3\alpha a_3 + (n-2-2\alpha)a_2 - (n-1)a_1| \\
&\quad + |2\alpha a_2 + (n-1-\alpha)a_1 - na_0| \\
&= |n\alpha a_n + a_{n-1}| + 2a_{n-2} - n\alpha a_n - (1+\alpha - n\alpha)a_{n-1} + 3a_{n-3} - (n-1)\alpha a_{n-1} \\
&\quad - (2+2\alpha - n\alpha)a_{n-2} + ... + (n-1)a_1 - 3\alpha a_3 - (n-2-2\alpha)a_2 \\
&\quad + na_0 - 2\alpha a_2 - (n-1-\alpha)a_1 \\
&= |n\alpha a_n + a_{n-1}| - n\alpha a_n - a_{n-1} + \alpha a_1 + na_0 \\
&= M'.
\end{aligned}$$

Since G(z) is analytic for $|z| \le 1$ and G(0)=0, it follows by Schwarz Lemma that $|G(z)| \le M|z|$ for $|z| \le 1$.

Hence, for $|z| \le 1$,

$$\begin{aligned}
|F(z)| &= |(\alpha a_1 + na_0) + G(z)| \\
&\ge |\alpha a_1 + na_0| - |G(z)| \\
&\ge |\alpha a_1 + na_0| - M'|z| \\
&> 0
\end{aligned}$$

if

$$|z| < \frac{|\alpha a_1 + na_0|}{M'}.$$

This shows that all the zeros of F(z) lie in $|z| \ge \frac{|\alpha a_1 + na_0|}{M'}$.

Since the zeros of $D_\alpha P(z)$ are also the zeros of F(z), it follows that all the zeros of $D_\alpha P(z)$ lie in $|z| \ge \frac{|\alpha a_1 + na_0|}{M'}$ and the proof of Theorem 3 is complete.

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