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*International Journal of Current Research*
*Vol. 8, Issue, 04, pp.29333-29340, April, 2016*

INTERNATIONAL JOURNAL OF CURRENT RESEARCH

ISSN: 0975-833X

RESEARCH ARTICLE

# A GRAPH THEORETIC SOLUTION TO THE 8 × 8 LIGHT'S OUT PROBLEM

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**ARTICLE INFO**

***Article History:***
Received 23rd January, 2016
Received in revised form
26th February, 2016
Accepted 09th March, 2016
Published online 26th April, 2016

***Key words:***

8x8 Light's Out Problem, Chasing the Light's out method.

**ABSTRACT**

This paper is concerned with a mathematical solution to the 8 × 8 Lights Out Problem, a modification of an originally 5 × 5 electronic one-person puzzle game played on a rectangular lattice of lamps which can be turned on and off. A move consists of flipping a "switch" inside one of the squares, thereby toggling the on/off state of this and all four vertically and horizontally adjacent squares. Starting from a randomly chosen light pattern, the aim is to turn all the lamps off. The researcher describes the mechanics of the game and the solution to it from two different perspectives, (the first as a curious puzzle fan, and the second as a mathematician), with the end intention of presenting a result of entertaining mathematical research and to share it with anyone who is interested in it.



**Citation: Edward M. Kiunisala, 2016.** "A graph theoretic solution to the 8 × 8 light's out problem", *International Journal of Current Research*, 8, (04), 29333-29340.

## 1. INTRODUCTION

Recreational mathematics is a treasury of problems which makes mathematics fun and which illustrates the idea that, "Mathematics is all around, one need only to look for it". It is used either as a diversion from serious mathematics or as a way of making serious mathematics understandable or palatable, and is already present in the oldest known mathematics and continue to the present day. Recreational questions are interspersed with more straightforward problems to provide breaks in the hard slog of learning. An additional utility of recreational mathematics is that it provides one a way to communicate mathematical ideas to the public at large. These problems are often based on reality, though with enough whimsy so that they have to be appealing to the students and the layman alike. The Lights Out Problem is an example of recreational mathematics. As such, a toy version of it, actually a handheld device, is made by Tiger Electronics, and is about the size of a VHS tape. It has 25 lights/buttons on it which can either be on or off. Whenever one hits a button, if that button is off, it turns on. If that button was on, it turns off. But the four buttons that are on the top, bottom, left and right of that button also switch states that way. The object is to get all of the lights off, hence the name of the game. Surprisingly, the Lights Out Problem is a mathematical problem in disguise, and a solution of it can be sought for if one is to approach the said problem from the said point of view. In this approach, one starts by checking to see if there are any lights still on in the top row. If there are, then the only way to turn them off without pressing any more buttons in the top row, is to press the buttons in the second row that are directly beneath the lit lights. One can't press any further buttons in the second row, since that would turn on lights in the top row. So now one can only use the bottom three rows. If one carries on in this way, he will eventually reach the bottom row. As can be seen, once one knows how to chase the lights to the bottom row, finding a solution amounts to tabulating all the possible combinations of lit squares at the bottom and finding the correct combination of lit squares to press on the top row that will eventually lead to each of the corresponding lit square combinations at the bottom.

Having an ample background on the underlying concepts of the original 5 × 5 Lights Out Problem and the method of Chasing the Lights, and realizing the potential of said problem in introducing the fundamentals of recreational mathematics, the researcher undertakes this study in an 8 × 8 grid, to emphasize his own personal conviction that mathematics can indeed, be fun and recreational.

## 2. Theoretical Background

Lights Out Problem involves toggling lights on and off. If a light is on, it must be toggled an odd number of times to be

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turned off. If a light is off, it must be toggled an even number of times (including not being toggled at all) for it to remain off. A successful operation is therefore a sequence of presses that toggles all the "on" lights an odd number of times and all the "off" lights an even number of times.

The following theorems are of utmost importance.

**Theorem 2.1 The order in which the lights are pressed does not matter.**

As an illustration, in the case of the original 5 × 5 grid, suppose the lights are numbered 1 to 25, left to right then from top to bottom. Pressing 3, 8 and 14 will toggle 2, 4, 7, 14, 15, and 19 exactly once while 3, 8, 9, and 13 will be toggled exactly twice, (see Figure 1 below). No matter in what order one presses 3, 8 and 14, all the affected lights will be toggled the same way in the end.

up will not necessarily be equal to the number of ways in which the pattern of lit squares in the bottom row of said grid can end up to after the method of Chasing the Lights is being applied. A classic example of this is the case $n = 5$, the original Lights Out Problem, it turns out!

For this purpose, let a $1 \times n$ vector $M$ be defined as an ordered $n$-tuple of numbers, $M = [m_1, m_2, \ldots, m_n]$ that lists the corresponding values of the squares in row $M$ of an $n \times n$ grid from left to right, (where a square is given a value of 1 if it is lit, otherwise it is given a value of 0), thus, Figure 2 below has a corresponding vector $M = [1\ 0\ 0\ 1\ 1]$.

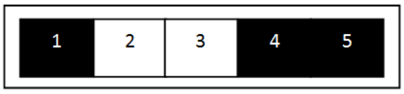


**Figure 2. A row with vector $M$ = [1 0 0 1 1]**

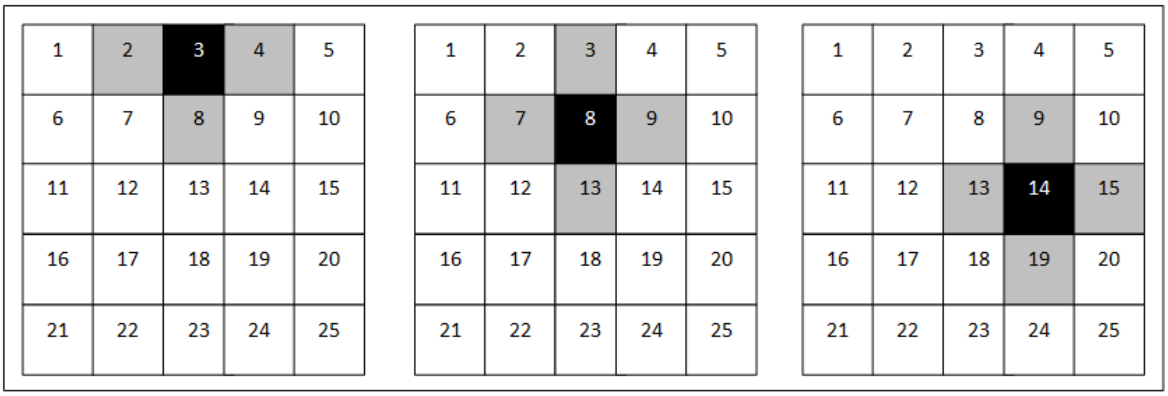


**Figure 1. Corresponding Lit On Squares When Squares 3, 8 and 14 Are Toggled Respectively**

**Theorem 2.2** In order to solve the Lights Out Problem, each light needs to be pressed no more than once.

Pressing a light an even number of times is equivalent to not touching it, and pressing it an odd number of times is equivalent to pressing it just once. Since the order in which the lights are pressed does not matter, a sequence in which one light is pressed an even number of times is equivalent to the same sequence with those even number of presses removed, and thus, the solution that uses the minimum number of moves is that in which no light is pressed more than once.

**Theorem 2.3** There are $2^n$ number of ways in which the squares at the top row of an $n \times n$ grid can be lit up.

Each of the squares at the top row of an 8 × 8 grid can be toggled on and off, and hence, the total number of possible ways in which the bottom row can be lit is $2 \times 2 \times \ldots \times 2$, (there are $n$ multiples of 2, since there are $n$ squares in the row).

**Theorem 2.4** The maximum number of ways in which the pattern of lit squares in the bottom row can end up to after the method of Chasing the Lights is being applied in an arbitrarily lit $n \times n$ grid is $2^n$.

It is of important interest that the number of ways in which the number of ways in which the top row of an $n \times n$ grid can be lit

At this juncture, let the five squares at the top row be denoted A through E, from left to right, while the squares at the bottom row be denoted 1 through 5, also from left to right, respectively. Furthermore, let an entry of 1 indicate that said square is lit, and that an entry of 0 indicates otherwise. In Figure 3 below, the right column indicates what squares at the bottom row of a 5 × 5 grid will remain lit if the corresponding square at the left column will be brought down using the method of chasing the lights.

In vector notation, the five starting lights, with each of their corresponding bottom row pattern can be written as thus:

A = [ 0 1 1 0 1 ]
B = [ 1 1 1 0 1 ]
C = [ 1 1 0 1 1 ]
D = [ 0 0 1 1 1 ]
E = [ 1 0 1 1 0 ]

The next concern now is to find out what particular pattern will the bottom row come out if a combination of two or more squares at the top row are simultaneously pressed before the method of chasing the lights is performed. By letting $n$ equals the number of squares at the top row and letting $r$ equals the number of squares that are pressed on said row before performing the method of chasing the lights, then the number of possible ways of pressing $r$ squares at a time from the set of

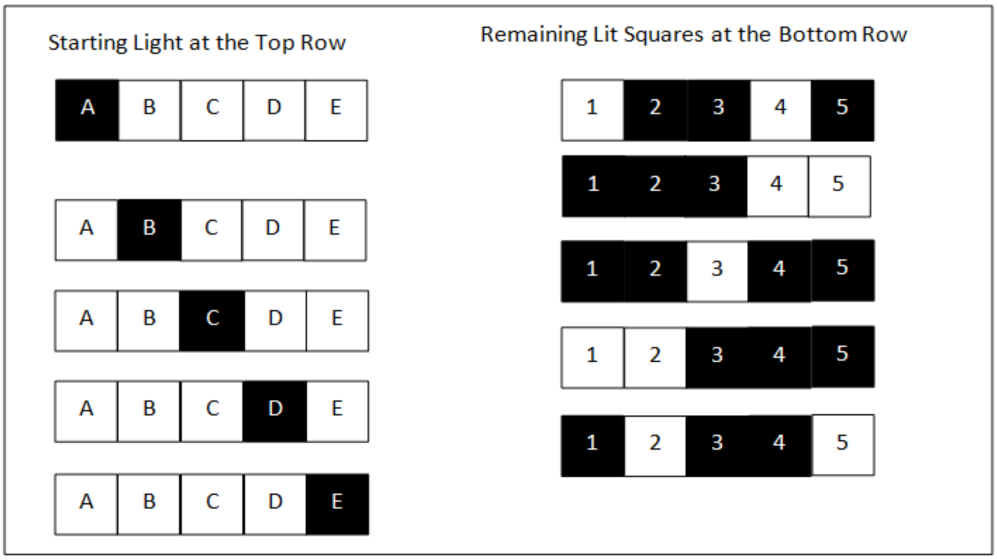


**Figure 3. What Squares at the Bottom Row Are Toggled When the Corresponding Squares at the Top Is Pressed**

$n$ available squares is given by the combination formula, herein denoted as is given by the formula, $\binom{n}{r} = \frac{n!}{r!(n-r)!}$, where $r! = r \cdot (r-1) \cdot (r-2) \cdot \ldots \cdot 3 \cdot 2 \cdot 1$. By definition, 0! = 1. Using the said combination formula, one expects that at the top row of a 5 × 5 grid, there are:

a) $\binom{5}{0} = \frac{5!}{0! \cdot 5!} = 1$ possible way of not pressing any square at a time

b) $\binom{5}{1} = \frac{5!}{1! \cdot 4!} = 5$ possible ways of pressing one square at a time

c) $\binom{5}{2} = \frac{5!}{2! \cdot 3!} = \frac{5 \cdot 4}{2!} = 10$ possible ways of pressing two squares at a time

d) $\binom{5}{3} = \frac{5!}{3! \cdot 2!} = \frac{5 \cdot 4}{2!} = 10$ possible ways of pressing three squares at a time

e) $\binom{5}{4} = \frac{5!}{4! \cdot 1!} = 5$ possible ways of pressing four squares at a time

f) $\binom{5}{5} = \frac{5!}{5! \cdot 0!} = 1$ possible way of pressing all the five squares at a time

All in all, there are 1 + 5 + 10+ 10+ 5 + 1 = 32 possible ways of pressing $r$ squares at a time from a set of 5 available squares at the top row of a 5 × 5 grid, which is what is to be expected as per Theorem 2.3 for $n$ = 5. Consider now the case of pressing two squares at the top row at a time before implementing the method of chasing the lights.

There are 10 such combinations, namely AB, AC, AD, AE, BC, BD, BE, CD, CE, and DE, respectively.

By virtue of Theorem 2.1 which states that the order in which the lights are pressed does not matter, pressing squares A and B, for example, is just equal to adding their corresponding vector notations. In this study and unless it will create a considerable confusion, the notation AB will stand to mean the vector sum A + B and is defined as:

$$A + B = AB = [a_1, a_2, \ldots, a_n] + [b_1, b_2, \ldots, b_n]$$

$$= [a_1+b_1, a_2+b_2, \ldots, a_n+b_n]$$
$$= [0\ 1\ 1\ 0\ 1] + [1\ 1\ 1\ 0\ 0]$$
$$= [1\ 2\ 2\ 0\ 1]$$

Invoking Theorem 2.2, however, which claims, among others, that pressing a light an even number of times is equivalent to not touching it, and pressing it an odd number of times is equivalent to pressing it just once, then any nonzero even entry of the vector sum AB can be rewritten as zero, and any odd entry on said vector sum can be rewritten as 1, and thus:

$$AB = [1\ \ 2\ \ 2\ \ 0\ \ 1] = [1\ \ 0\ \ 0\ \ 0\ \ 1]$$

Figure 4 below summarizes as to what particular pattern will the bottom row come out if a combination of two or more squares at the top row are simultaneously pressed before the method of chasing the lights is performed.

As what the table indicates, pressing square A on top will result to the lighting of squares 2, 3, and 5 in the bottom row. The same result is obtained if one is to start by pressing squares C and E and chasing the lights all the way to the last bottom. To this effect, observe here that the $2^5$ = 32 possible ways in which the top row can be lit up resulted only to 8 possible ways, (including the trivial all lights out solution), in which the pattern of lit squares in the bottom row can end up to after the method of Chasing the Lights.

| Squares to be Toggled at the Top Row | Remaining Lit Squares At the Bottom Row |
|---|---|
| A, CE, BDE, ABCD | 01101 |
| B, CD, ADE, ABCE | 11100 |
| C, AE, BD, ABCDE | 11011 |
| D, BC, ACDE, ABE | 00111 |
| E, AC, ABD, BCDE | 10110 |
| AB, DE, ACD, BCE | 10001 |
| ACE, BCD, ABDE, (not a square toggled) | 00000 |
| AD, BE, ABC, CDE | 01010 |

**Figure 4. A Table That Indicates What Squares at the Bottom Row Will Light Up If the Corresponding Squares at the Top Row Are Toggled**

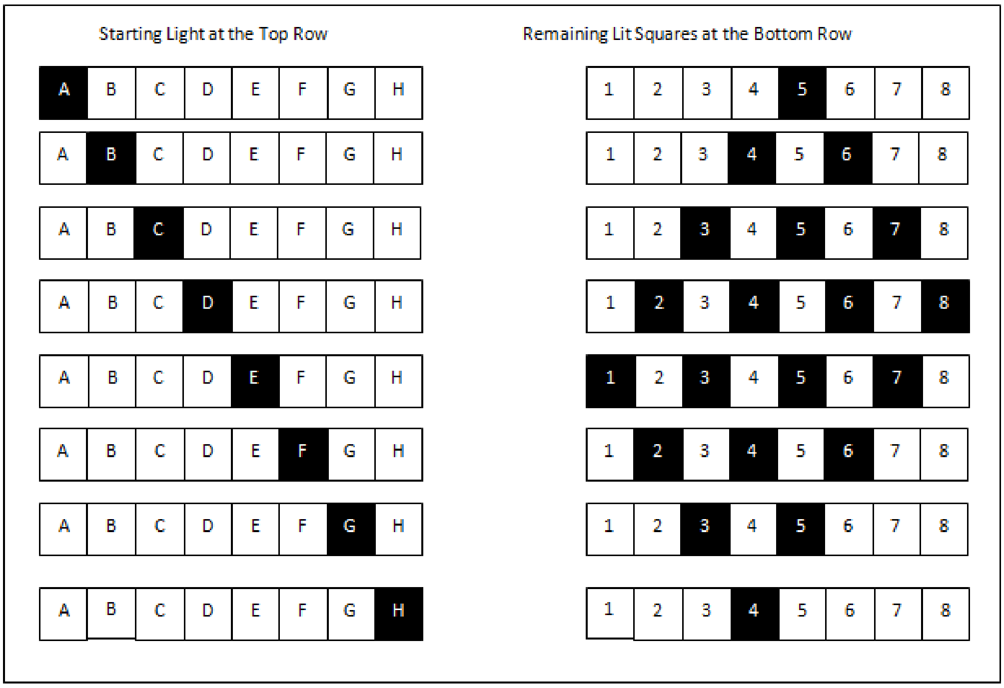


**Figure 5. Table Showing What Squares at the Bottom Row Are Toggled When the Corresponding Squares at the Top Is Pressed**

**In the vector notation, it can thus be said that the eight basic vectors of an 8 × 8 grid are the following:**

$$A = [0\ \ 0\ \ 0\ \ 0\ \ 1\ \ 0\ \ 0\ \ 0] \qquad E = [1\ \ 0\ \ 1\ \ 0\ \ 1\ \ 0\ \ 1\ \ 0]$$

$$B = [0\ \ 0\ \ 0\ \ 1\ \ 0\ \ 1\ \ 0\ \ 0] \qquad F = [0\ \ 1\ \ 0\ \ 1\ \ 0\ \ 1\ \ 0\ \ 0]$$

$$C = [0\ \ 0\ \ 1\ \ 0\ \ 1\ \ 0\ \ 1\ \ 0] \qquad G = [0\ \ 0\ \ 1\ \ 0\ \ 1\ \ 0\ \ 0\ \ 0]$$

$$D = [0\ \ 1\ \ 0\ \ 1\ \ 0\ \ 1\ \ 0\ \ 1] \qquad H = [0\ \ 0\ \ 0\ \ 1\ \ 0\ \ 0\ \ 0\ \ 0]$$

## 3 CHASING THE LIGHTS ON AN 8 × 8 GRID

The most common method to solve the Lights Out Problem is to start by wiping all the lights except for in the last or bottom row. This is done by pressing lights that are directly below lights that are turned on to cancel them out until only lights in the last row remain. In the case of the 8 × 8 grid, there are 8 possible ways to light exactly one square at a time at the top row. The right column in Figure 5 below indicates what squares will remain lit at the bottom row if the corresponding square at the left column will be brought down using the method of chasing the lights. Labeling the squares in the first row from left to right using the letters A through H, and doing the same in the last row using the numbers 1 through 8, this table indicates that by pressing square A in the top row and following the steps to bring down all the lights to the bottom row, then square 5 in the bottom row will be toggled. The next step now is to find all the possible combinations of selecting two or more squares to be pressed on the top row and find out what pattern at the bottom row will each of the above combinations end up to. Using the combination formula with *n*

= 8 and $r$ ranging from 0 to 8, one finds that at the top row of an 8 × 8 grid, there are:

a) $\binom{8}{0} = \frac{8!}{0!\cdot 8!} = 1$ possible way of not pressing any square at a time

b) $\binom{8}{1} = \frac{8!}{1!\cdot 7!} = 8$ possible ways of pressing one square at a time

c) $\binom{8}{2} = \frac{8!}{2!\cdot 6!} = \frac{8\cdot 7}{2} = 28$ possible ways of pressing two squares at a time

d) $\binom{8}{3} = \frac{8!}{3!\cdot 5!} = \frac{8\cdot 7\cdot 6}{3!} = 56$ possible ways of pressing three squares at a time

e) $\binom{8}{4} = \frac{8!}{4!\cdot 4!} = \frac{8\cdot 7\cdot 6\cdot 5}{4!} = 70$ possible ways of pressing four squares at a time

f) $\binom{8}{5} = \frac{8!}{5!\cdot 3!} = \frac{8\cdot 7\cdot 6}{3!} = 56$ possible ways of pressing five squares at a time

g) $\binom{8}{6} = \frac{8!}{6!\cdot 2!} = \frac{8\cdot 7}{2} = 28$ possible ways of pressing six squares at a time

h) $\binom{8}{7} = \frac{8!}{7!\cdot 1!} = 8$ possible ways of pressing seven squares at a time

i) $\binom{8}{8} = \frac{8!}{8!\cdot 0!} = 1$ possible way of pressing all the eight squares at a time

This makes a total of 1 + 8 + 28 + 56 + 70 + 56 + 28 + 8 + 1 = 256 possible ways of pressing $r$ squares at a time from a set of 8 available squares at the top row, which is in tally with Theorem 2.3 for $n = 8$.

The main concern of this study is to investigate if there is a solution to each and every possible pattern that the remaining lit switches at the bottom row can have after the first round of Chasing the Lights for any initial pattern is being made, thereby identifying along the way what squares to press at the top row to come up with the desired pattern at the bottom row. In the light of this objective, the researcher switches the rolls of the top row and the bottom row, that is, by listing all the possible light configurations at the bottom row and by the use of vector addition of the eight basic vectors taken $r$ at a time, (where $r$ ranges from 0 to 8), a correspondence between the two listings is then made to figure out what particular square combinations to be lit at the top row will end up to what particular possible light pattern in the bottom row.

There are $\binom{8}{2} = \frac{8!}{2!\cdot 6!} = \frac{8\cdot 7}{2} = 28$ possible ways that the bottom row can end up with two lit squares.

**A solution to each of said possible ways is tabulated below:**

**Table 1. Solutions to 2 lights on at the bottom row**

| SEQUENCE | | | | | | | | SOLUTION | SOLUTION | SEQUENCE | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | | | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | BCEF | CDFG | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | ACEG | BDFH | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 |
| 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | CEH | ADF | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 |
| 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | ABFG | BCGH | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 |
| 0 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | BFH | ACG | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 0 |
| 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | ACE | DFH | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 |
| 0 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | G | B | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 0 |
| 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | BCEH | ADFG | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 |
| 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | EG | BD | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 |
| 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | ABF | CGH | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 0 |
| 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | FH | AC | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 |
| 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | AGH | ABH | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 |
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | CDEF | ABGH | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 |
| 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | AH | BCFG | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 |

**Table 2. Solutions to 3 lights on at the bottom row**

| SEQUENCE | | | | | | | | SOLUTION | SOLUTION | SEQUENCE | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | | | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | ABCEFG | BCDFGH | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 |
| 1 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | BCEFH | ACDFG | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 1 |
| 1 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | ACEGH | ABDFH | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 1 |
| 1 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | ABCEF | CDFGH | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 1 |
| 0 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | ABFGH | ABCGH | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 0 |
| 1 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | CEFH | ACDF | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 1 |
| 1 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | CEG | BDF | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 1 |
| 1 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | BEFG | BCDG | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 1 |

*Continue…..*

| | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 1 | 0 | 1 | 0 | 0 | 0 | BFG | BCG | 0 | 0 | 0 | 1 | 0 | 1 | 1 | 0 |
| 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | ACEH | ADFH | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 1 |
| 1 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | BCDE | DEFG | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 0 | 1 | 0 | 1 | 1 | 0 | 0 | 0 | ABFH | ACGH | 0 | 0 | 0 | 1 | 1 | 0 | 1 | 0 |
| 1 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | ABCEGH | ABDFGH | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 1 |
| 0 | 1 | 1 | 0 | 0 | 1 | 0 | 0 | AFGH | ABCH | 0 | 0 | 1 | 0 | 0 | 1 | 1 | 0 |
| 1 | 0 | 0 | 1 | 0 | 1 | 0 | 0 | BCE | DFG | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 1 |
| 1 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | EGH | ABD | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 1 |
| 1 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | CDEFH | ACDEF | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 1 |
| 0 | 1 | 0 | 1 | 0 | 1 | 0 | 0 | F | C | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| 1 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | BEGH | ABDG | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 1 |
| 0 | 0 | 1 | 1 | 0 | 1 | 0 | 0 | ABG | BGH | 0 | 0 | 1 | 0 | 1 | 1 | 0 | 0 |
| 1 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | BCDEFG | ACDEFG | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 1 |
| 1 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | AE | DH | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 1 |
| 1 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | ABCEH | ADFGH | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 1 |
| 0 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | AFH | ACH | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 0 |
| 1 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | AEG | BDH | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 1 |
| 0 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | AB | GH | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 0 |
| 0 | 1 | 1 | 0 | 0 | 0 | 1 | 0 | ABCF | CFGH | 0 | 1 | 0 | 0 | 0 | 1 | 1 | 0 |
| 0 | 1 | 0 | 1 | 0 | 0 | 1 | 0 | BCFGH | ABCFG | 0 | 1 | 0 | 0 | 1 | 0 | 1 | 0 |

**Table 3. Solutions to 4 lights on at the bottom row**

| SEQUENCE | | | | | | | | SOLUTION | SOLUTION | SEQUENCE | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | | | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | ABCEFGH | ABCDFGH | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 |
| 1 | 1 | 0 | 0 | 0 | 1 | 1 | 0 | EFGH | ABCD | 0 | 1 | 1 | 0 | 0 | 0 | 1 | 1 |
| 1 | 1 | 1 | 0 | 1 | 0 | 0 | 0 | BCEFG | BCDFG | 0 | 0 | 0 | 1 | 0 | 1 | 1 | 1 |
| 1 | 1 | 0 | 1 | 1 | 0 | 0 | 0 | ABCEFH | ACDFGH | 0 | 0 | 0 | 1 | 1 | 0 | 1 | 1 |
| 1 | 1 | 1 | 0 | 0 | 1 | 0 | 0 | ACEFGH | ABCDFH | 0 | 0 | 1 | 0 | 0 | 1 | 1 | 1 |
| 1 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | CEGH | ABDF | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 1 |
| 1 | 1 | 1 | 0 | 0 | 0 | 1 | 0 | ABEF | CDGH | 0 | 1 | 0 | 0 | 0 | 1 | 1 | 1 |
| 0 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | ABCG | BFGH | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 0 |
| 1 | 1 | 0 | 1 | 0 | 1 | 0 | 0 | CEF | CDF | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 1 |
| 1 | 1 | 1 | 0 | 0 | 0 | 0 | 1 | ABCDEG | BDEFGH | 1 | 0 | 0 | 0 | 0 | 1 | 1 | 1 |
| 1 | 1 | 0 | 1 | 0 | 0 | 1 | 0 | BEFGH | ABCDG | 0 | 1 | 0 | 0 | 1 | 0 | 1 | 1 |
| 1 | 0 | 1 | 1 | 0 | 1 | 0 | 0 | ABCEG | BDFGH | 0 | 0 | 1 | 0 | 1 | 1 | 0 | 1 |
| 1 | 1 | 0 | 1 | 0 | 0 | 0 | 1 | BCDEH | ADEFG | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 1 |
| 0 | 1 | 1 | 1 | 0 | 1 | 0 | 0 | AFG | BCH | 0 | 0 | 1 | 0 | 1 | 1 | 1 | 0 |
| 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | ACEFH | ACDFH | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 |
| 1 | 0 | 1 | 0 | 1 | 1 | 0 | 0 | BCEGH | ABDFG | 0 | 0 | 1 | 1 | 0 | 1 | 0 | 1 |
| 1 | 1 | 0 | 0 | 1 | 0 | 1 | 0 | ABEFG | BCDGH | 0 | 1 | 0 | 1 | 0 | 0 | 1 | 1 |
| 0 | 1 | 1 | 0 | 1 | 1 | 0 | 0 | FGH | ABC | 0 | 0 | 1 | 1 | 0 | 1 | 1 | 0 |
| 1 | 1 | 0 | 0 | 1 | 0 | 0 | 1 | ABCDE | DEFGH | 1 | 0 | 0 | 1 | 0 | 0 | 1 | 1 |
| 1 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | ABCE | DFGH | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 1 |
| 1 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | CDEH | ADEF | 1 | 0 | 1 | 0 | 0 | 0 | 1 | 1 |
| 0 | 1 | 0 | 1 | 1 | 1 | 0 | 0 | AF | CH | 0 | 0 | 1 | 1 | 1 | 0 | 1 | 0 |
| 1 | 0 | 1 | 1 | 0 | 0 | 1 | 0 | AEH | ADH | 0 | 1 | 0 | 0 | 1 | 1 | 0 | 1 |
| 0 | 1 | 1 | 1 | 0 | 0 | 1 | 0 | ABCFH | ACFGH | 0 | 1 | 0 | 0 | 1 | 1 | 1 | 0 |
| 1 | 0 | 0 | 1 | 1 | 0 | 1 | 0 | AEGH | ABDH | 0 | 1 | 0 | 1 | 1 | 0 | 0 | 1 |
| 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | E | D | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| 1 | 0 | 1 | 1 | 0 | 0 | 0 | 1 | ACDEFGH | ABCDEFH | 1 | 0 | 0 | 0 | 1 | 1 | 0 | 1 |
| 1 | 0 | 1 | 0 | 1 | 0 | 0 | 1 | CDEFG | BCDEF | 1 | 0 | 0 | 1 | 0 | 1 | 0 | 1 |
| 1 | 0 | 1 | 0 | 0 | 1 | 1 | 0 | ABEH | ADGH | 0 | 1 | 1 | 0 | 0 | 1 | 0 | 1 |
| 1 | 0 | 0 | 1 | 0 | 1 | 1 | 0 | BEG | BDG | 0 | 1 | 1 | 0 | 1 | 0 | 0 | 1 |
| 0 | 1 | 1 | 0 | 1 | 0 | 1 | 0 | BCF | CFG | 0 | 1 | 0 | 1 | 0 | 1 | 1 | 0 |
| 1 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | ABEGH | ABDGH | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 1 |
| 0 | 0 | 1 | 1 | 1 | 1 | 0 | 0 | BG | ACFH | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 |
| 1 | 1 | 0 | 0 | 0 | 0 | 1 | 1 | BDEG | ACDEFH | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 1 |
| 1 | 0 | 1 | 0 | 0 | 1 | 0 | 1 | ABCDEFGH | ABCFGH | 0 | 1 | 0 | 1 | 1 | 0 | 1 | 0 |

**Table 4. Solutions to 5 lights on at the bottom row**

| SEQUENCE | | | | | | | | SOLUTION | SOLUTION | SEQUENCE | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | | | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | BCEFGH | ABCDFG | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 |
| 1 | 1 | 1 | 1 | 0 | 1 | 0 | 0 | ACEFG | BCDFH | 0 | 0 | 1 | 0 | 1 | 1 | 1 | 1 |
| 1 | 1 | 1 | 1 | 0 | 0 | 1 | 0 | ABEFH | ACDGH | 0 | 1 | 0 | 0 | 1 | 1 | 1 | 1 |
| 1 | 1 | 1 | 0 | 1 | 1 | 0 | 0 | CEFGH | ABCDF | 0 | 0 | 1 | 1 | 0 | 1 | 1 | 1 |
| 1 | 1 | 1 | 1 | 0 | 0 | 0 | 1 | ABDEFGH | ABCDEGH | 1 | 0 | 0 | 0 | 1 | 1 | 1 | 1 |
| 1 | 1 | 0 | 1 | 1 | 1 | 0 | 0 | ACEF | CDFH | 0 | 0 | 1 | 1 | 1 | 0 | 1 | 1 |
| 1 | 1 | 1 | 0 | 1 | 0 | 1 | 0 | BEF | CDG | 0 | 1 | 0 | 1 | 0 | 1 | 1 | 1 |
| 1 | 0 | 1 | 1 | 1 | 1 | 0 | 0 | BCEG | BDFG | 0 | 0 | 1 | 1 | 1 | 1 | 0 | 1 |
| 1 | 1 | 1 | 0 | 1 | 0 | 0 | 1 | BCDEG | BDEFG | 1 | 0 | 0 | 1 | 0 | 1 | 1 | 1 |
| 0 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | FG | BC | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 0 |
| 1 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | AEFH | ACDH | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 1 |
| 1 | 1 | 0 | 1 | 1 | 0 | 1 | 0 | ABEFGH | ABCDGH | 0 | 1 | 0 | 1 | 1 | 0 | 1 | 1 |
| 1 | 1 | 1 | 0 | 0 | 1 | 0 | 1 | ACDEGH | ABDEFH | 1 | 0 | 1 | 0 | 0 | 1 | 1 | 1 |
| 1 | 0 | 1 | 1 | 1 | 0 | 1 | 0 | EH | AD | 0 | 1 | 0 | 1 | 1 | 1 | 0 | 1 |
| 1 | 1 | 1 | 0 | 0 | 0 | 1 | 1 | ABDE | DEGH | 1 | 1 | 0 | 0 | 0 | 1 | 1 | 1 |
| 0 | 1 | 1 | 1 | 1 | 0 | 1 | 0 | BCFH | ACFG | 0 | 1 | 0 | 1 | 1 | 1 | 1 | 0 |
| 1 | 1 | 0 | 1 | 1 | 0 | 0 | 1 | ABCDEH | ADEFGH | 1 | 0 | 0 | 1 | 1 | 0 | 1 | 1 |
| 1 | 1 | 0 | 1 | 0 | 1 | 1 | 0 | EFG | BCD | 0 | 1 | 1 | 0 | 1 | 0 | 1 | 1 |
| 1 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | CDE | DEF | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 1 |
| 1 | 0 | 1 | 1 | 0 | 1 | 1 | 0 | ABE | DGH | 0 | 1 | 1 | 0 | 1 | 1 | 0 | 1 |
| 1 | 1 | 0 | 1 | 0 | 0 | 1 | 1 | BDEGH | ABDEG | 1 | 1 | 0 | 0 | 1 | 0 | 1 | 1 |
| 1 | 1 | 0 | 0 | 1 | 1 | 1 | 0 | AEFGH | ABCDH | 0 | 1 | 1 | 1 | 0 | 0 | 1 | 1 |
| 0 | 1 | 1 | 1 | 0 | 1 | 1 | 0 | ACF | CFH | 0 | 1 | 1 | 0 | 1 | 1 | 1 | 0 |
| 1 | 0 | 1 | 0 | 1 | 1 | 1 | 0 | BEH | ADG | 0 | 1 | 1 | 1 | 0 | 1 | 0 | 1 |
| 1 | 0 | 0 | 1 | 1 | 1 | 1 | 0 | ABEG | BDGH | 0 | 1 | 1 | 1 | 1 | 0 | 0 | 1 |
| 1 | 0 | 1 | 1 | 1 | 0 | 0 | 1 | ABCDEF | CDEFGH | 1 | 0 | 0 | 1 | 1 | 1 | 0 | 1 |
| 1 | 1 | 0 | 0 | 1 | 1 | 0 | 1 | ACDEH | ADEFH | 1 | 0 | 1 | 1 | 0 | 0 | 1 | 1 |
| 1 | 0 | 1 | 1 | 0 | 1 | 0 | 1 | ABCDEFG | BCDEFGH | 1 | 0 | 1 | 0 | 1 | 1 | 0 | 1 |

**Table 5. Solutions to 6 lights on at the bottom row**

| SEQUENCE | | | | | | | | SOLUTION | SOLUTION | SEQUENCE | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | | | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 1 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | CEFG | BCDF | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1 | 1 | 1 | 1 | 1 | 0 | 1 | 0 | BEFH | ACDG | 0 | 1 | 0 | 1 | 1 | 1 | 1 | 1 |
| 1 | 1 | 1 | 1 | 1 | 0 | 0 | 1 | BCDEGH | ABDEFG | 1 | 0 | 0 | 1 | 1 | 1 | 1 | 1 |
| 1 | 1 | 1 | 1 | 0 | 1 | 1 | 0 | AEF | CDH | 0 | 1 | 1 | 0 | 1 | 1 | 1 | 1 |
| 1 | 1 | 1 | 1 | 0 | 1 | 0 | 1 | ACDEG | BDEFH | 1 | 0 | 1 | 0 | 1 | 1 | 1 | 1 |
| 1 | 1 | 1 | 0 | 1 | 1 | 1 | 0 | EFH | ACD | 0 | 1 | 1 | 1 | 0 | 1 | 1 | 1 |
| 1 | 1 | 1 | 1 | 0 | 0 | 1 | 1 | ABDEH | ADEGH | 1 | 1 | 0 | 0 | 1 | 1 | 1 | 1 |
| 1 | 1 | 0 | 1 | 1 | 1 | 1 | 0 | AEFG | BCDH | 0 | 1 | 1 | 1 | 1 | 0 | 1 | 1 |
| 1 | 1 | 1 | 0 | 1 | 1 | 0 | 1 | CDEGH | ABDEF | 1 | 0 | 1 | 1 | 0 | 1 | 1 | 1 |
| 1 | 0 | 1 | 1 | 1 | 1 | 1 | 0 | BE | DG | 0 | 1 | 1 | 1 | 1 | 1 | 0 | 1 |
| 1 | 1 | 1 | 0 | 1 | 0 | 1 | 1 | BDE | DEG | 1 | 1 | 0 | 1 | 0 | 1 | 1 | 1 |
| 1 | 1 | 0 | 1 | 1 | 1 | 0 | 1 | ACDE | DEFH | 1 | 0 | 1 | 1 | 1 | 0 | 1 | 1 |
| 1 | 1 | 1 | 0 | 0 | 1 | 1 | 1 | ADEH | BCDEFG | 1 | 0 | 1 | 1 | 1 | 1 | 0 | 1 |
| 1 | 1 | 0 | 1 | 1 | 0 | 1 | 1 | ABDEGH | CF | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 0 |

**Table 6. Solutions to 7 lights on at the bottom row**

| SEQUENCE | | | | | | | | SOLUTION | SOLUTION | SEQUENCE | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | | | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | EF | CD | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1 | 1 | 1 | 1 | 1 | 1 | 0 | 1 | CDEG | BDEF | 1 | 0 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1 | 1 | 1 | 1 | 1 | 0 | 1 | 1 | BDEH | ADEG | 1 | 1 | 0 | 1 | 1 | 1 | 1 | 1 |
| 1 | 1 | 1 | 1 | 0 | 1 | 1 | 1 | ADE | DEH | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 1 |

The combination formula shows that there is only $\binom{8}{8} = 1$ way that all squares in the bottom row are lit. The unique combination of squares D and E at the top row will deliver the desired effect.

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