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RESEARCH ARTICLE

REVIEW OF COMPLEX CONFORMAL STRUCTURES OF 1-DIMENSIONAL RIEMANNIAN MANIFOLD

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ABSTRACT

The purpose of this paper is to give an idea of 1-dimensional Riemannian manifolds, and their complex structure. We are introducing complex structure, Riemannian surface, equivalence of conformal and complex structure, complex structure generated by metric, Beltrami coefficient, Beltrami equation, Cauchy-Kowalewski Theorem, Classification of Riemann surfaces, moduli of Riemann surfaces, associated hermitian form, Riemann surface with Riemann metric g compatible with the almost complex structure, the exceptional Riemann Surfaces; theorems and proposition related them will be presented with and without proof.

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INTRODUCTION

Eisenhart (1927) gave the theory of conformal structures arose in studying those properties of Riemannian and pseudo-Riemannian manifolds that remain invariant under conformal transformations of the metric. The problem of conformal coordinates was studied by Gauss, who proved their existence in the real-analytic case. Josh Guffin (2004) gave the Definition of Riemann surface as: A complex manifold of complex dimension 1 is called a Riemann surface or 1-Dimensional Riemannian manifold and introduce a Complex structure as: Suppose that $\mathcal{M}$ is a 2n-dimensional manifold, having one atlas $\{\mathcal{A}_{\mathrm{a}}, \mathcal{F}_{\mathrm{a}}\}$; $\cup_{\mathrm{a}}\mathcal{A}_{\mathrm{a}} = \mathcal{M}, \mathcal{F}_{\mathrm{a}}: \mathcal{A}_{\mathrm{a}} \longrightarrow \mathbb{C}^{\mathrm{n}}$. If the functions $\mathcal{F}_{\mathrm{a}} \circ \mathcal{F}_{\mathrm{b}}^{-1}: \mathbb{C}^{\mathrm{n}} \longrightarrow \mathbb{C}^{\mathrm{n}}$. are holomorphic on their DOD (domains of definition), the atlas is known as complex-analytic. Two atlases $\{\mathcal{A}_{\mathrm{a}}, \mathcal{F}_{\mathrm{a}}\}$ & $\{\mathcal{B}_{\mathrm{a}}, \acute{\mathcal{F}}_{\mathrm{a}}\}$ are called compatible if their union is again an atlas. Clearly this defines an equivalence relation on the set of atlases, an equivalence class of which is known as a complex structure. A manifold together with complex structure is known as complex manifold. Its complex dimension is defined to be n; $dim\mathbb{C}^n = n$ .Other way to define a complex structure is to complexify the tangent bundle and introduce an almost-complex structure J. An almost-complex structure on $\mathcal{M}$ is a tensor of type (1, 1) which squares to −1. In local coordinates this is

$$J_b^a J_c^b = -\delta_c^a.$$

An almost complex structure J is called to be integrable if the Nijenhuis tensor

$$N(X,Y) = [X,Y] + J[JX,Y] + J[X,JY] - [JX,JY]$$

vanishes for all smooth vector fields X and Y.

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An integrable almost-complex structure defines a complex structure on a manifold. In one complex dimension, the Nijenhuis tensor $N(X,Y) = (X,Y) + J(JX,Y) + J(X,JY) - (JX,JY)$ vanishes identically, so that every almost-complex structure defines a complex structure. Any complex structure has a unique associated almost-complex structure.

Almost Complex Structures: An almost complex structure on X is a map

$J : \mathcal{A}^1_{X,\square} \longrightarrow \mathcal{A}^1_{X,\square}$

of $\mathcal{A}_{X,\square}$ modules, such that $J^2 = -1$.

We may $\square$-linearly extend J to $\mathcal{A}^1_X$, and thus obtain decomposition

$\mathcal{A}^1_X = \mathcal{A}^{1,0}_X \oplus \mathcal{A}^{0,1}_X$

where J acts as multiplication with i on $\mathcal{A}^{1,0}_X$, and as $-\ i$ on $\mathcal{A}^{0,1}_X$. Moreover,

$\mathcal{A}^{1,0}_X = \mathcal{A}^{\overline{0,1}}_X$

Remark 1.1 There is a one to one correspondence between almost complex structures on X and decompositions of differentiable vector bundles

$\mathcal{A}^1_X = \mathcal{A}^{1,0}_X \oplus \mathcal{A}^{0,1}_X$ s.t.

$\mathcal{A}^{\overline{0,1}}_X = \mathcal{A}^{1,0}_X$.

2. Almost complex structure of a Riemann surface

If X is a Riemann surface it means there is a natural almost complex structure. For given coordinates x, y we define

$$J\ dx := -\ dy, J(dy) = dx. \tag{1}$$

In order to see that this is independent of the choice of the coordinates let

$f = f_1\ x,y\ + if_2(x,y)$

be a holomorphic function and $f_1\ x,y$, $f_2(x,y)$ another choice of coordinates. We compute

$J\ df_1 = J\ \frac{\partial f_1}{\partial x}dx + \frac{\partial f_1}{\partial y}dy$	By Partial Differentiation	
$= \frac{\partial f_1}{\partial x}\mathrm{J\ dx} + \frac{\partial f_1}{\partial y}\mathrm{J\ dy}$	By Partial Differentiation property	
$= -\frac{\partial f_1}{\partial x}dy + \frac{\partial f_1}{\partial y}dx$	Using Expression (1)	(2)
$= -\frac{\partial f_2}{\partial y}dy - \frac{\partial f_2}{\partial x}dx$	(By Cauchy-Riemann Condition)	
$= -df_2.$	By Partial Differentiation	

and

$J\ df_2 = J\ \frac{\partial f_2}{\partial x}dx + \frac{\partial f_2}{\partial y}dy$	By Partial Differentiation
$= \frac{\partial f_2}{\partial x}\mathrm{J\ dx} + \frac{\partial f_2}{\partial y}\mathrm{J\ dy}$	By Partial Differentiation property
$= -\frac{\partial f_2}{\partial x}dy + \frac{\partial f_2}{\partial y}dx$	Using Expression (1)
$= \frac{\partial f_1}{\partial y}dy + \frac{\partial f_1}{\partial x}dx$	By Cauchy-Riemann Condition
$= -df_1.$	By Partial Differentiation

Thus 1 is well-defined, and defines an almost complex structure on X.

Obviously, $J\ dz = idz, J\ d\bar{z} = -id\bar{z}$, so that $\mathcal{A}^{1,0}_X$ is the eigenspace of J with eigenvalues i, and $\mathcal{A}^{0,1}_X$ is the eigenspace of J with eigenvalue $-\ i$. Note that $\mathcal{A}^{1,0}_X$ (resp.$\mathcal{A}^{1,0}_X$) defines J uniquely.

Remark 2.1 Let X, Y be Riemann surfaces, and $f: X \longrightarrow Y$ a map of differentiable manifolds. Show that f is holomorphic iff f^* commutes with J, i.e. the following diagram commutes

$$\begin{tikzcd} \mathcal{A}_Y^1 \arrow[r, "f^*"] \arrow[d, "J"] & \mathcal{A}_X^1 \arrow[d, "J"'] \\ \mathcal{A}_Y^1 \arrow[r, "f^*"] & \mathcal{A}_X^1 \end{tikzcd}$$

Theorem of Newlander-Nirenberg: Let CI be the category of differentiable 1-complex-dimensional manifolds equipped with an almost complex structure. The morphisms of CI are differentiable maps which commute with J. Remark 2.1 Yields a conclusion:

$$\{\text{Riemann surfaces}\} \rightarrow \text{CI} \tag{3}$$

Remark 2.1, shows that these conclusions for two Riemann surfaces X, Y the map

$\mathrm{Hom}_{\mathrm{RS}}\ \mathrm{X}, \mathrm{Y} \ \longrightarrow \mathrm{Hom}_{\mathrm{CI}}\ \mathrm{X}, \mathrm{Y}$ is bijective.

Theorem 2.1 The conclusion (3) is an equivalence of categories.

Proof: We will show that every object in CI comes from a Riemann surface. If X is a Riemann surface and z a local coordinate then $dz \in \mathcal{A}_X^{1,0}$ and therefore the $(0,1)$ -part of dz vanishes.

Suppose X is a 2-D differentiable manifold and J an almost complex structure on X. For every function $f \in \mathcal{A}_X^1$ we may decompose $df = \omega^{1,0} + \omega^{0,1}$. The Theorem of Newlander-Nirenberg asserts that locally (say at $x \in X$) we can find a function f such that $\omega^{0,1} = 0$, and $\omega^{1,0}(x) \neq 0$. Such a function f defines a diffeomorphism

$$f : U_{(x,f)} \longrightarrow \Delta_r$$

for a suitable neighbourhood $U_{(x,f)}$ of x, and Δ_r the open disc with radius r (for suitable r). Moreover, the map f commutes with the almost complex multiplication. $Jf^* = f^* J\Delta_r$.

Let $y \in X$ be another point and g a function around y with $(dg)^{0,1} = 0$. The induced map

$$f\ U_{y,g} \cap U_{x,f} \overset{\cong}{\rightarrow} g(U_{(y,g)} \cap U_{x,f})$$

is holomorphic, because it commutes with J. Therefore we obtain a complex atlas for X.

Metric and Curvature on Riemann Surface

Suppose X is a manifold. We write $\mathcal{A}_{X,\mathbb{R}}$ for the sheaf of real functions, and $\mathcal{A}_{X,\mathbb{R}}^1$, for the sheaf of real 1-forms.
A Riemann metric g on $\mathcal{A}_{X,\mathbb{R}}^1$ is a symmetric pairing of $\mathcal{A}_{X,\mathbb{R}}$ -modules

$$g : \mathcal{A}_{X,\mathbb{R}}^1 \times \mathcal{A}_{X,\mathbb{R}}^1 \longrightarrow \mathcal{A}_{X,\mathbb{R}},$$

with the following property. For all local section s $\in \mathcal{A}_{X,\mathbb{R}}^1$ defined at a point

$$x \in X: g\ s, s\ \ x \ \geq 0 \text{ and } g(s,s)(x) = 0 \Leftrightarrow s(x) = 0.$$

For the existence of a Riemann metric we note that the sum of two metrics and the multiple of a metric by a positive function is again a metric. Since locally (i.e. for open sets in $\mathbb{R}$) there is a metric we can patch the local metrics together by using partition of unity.

Remark 3.1 Usually a metric on the tangent bundle $T_{X,\mathbb{R}}$ which is dual to the differential one form $T_{X,\mathbb{R}}^V = \mathcal{A}_{X,\mathbb{R}}^1$,

- There is a canonical correspondence between the metrics of a vector space and its dual.
- The same applies to our situation.

Proposition 3.1 Suppose X is a 2-D oriented manifold with a Riemann metric g. There is a unique almost complex structure J on X such that $g(Js, Jt) = g(s,t)$ for all sections s, t, and J preserves the orientation.

Proof: Uniqueness: We check the uniqueness locally (since J is a map of sheaves). Say we choose coordinates x, y such that $dx \wedge dy$ is positively oriented. By the Gram-Schmidt process we find an orthonormal base s, t of $\mathcal{A}^1_{X,?}$. Furthermore we may suppose that $s \wedge t$ is positively oriented, i.e. $s \wedge t = f dx \wedge dy$ with $f > 0$. Write $J(s) = as + ct, J(t) = bs + dt$, then $a^2 + c^2 = 1 = b^2 + d^2$ and $ab + cd = 0$, and thus

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & -c \\ c & a \end{pmatrix} \text{ or } \begin{pmatrix} a & c \\ c & -a \end{pmatrix}$$

Since J preserves the orientation we get $J(s) \wedge J(t) = (ad - bc) s \wedge t$ with $ad - bc > 0$ and thus

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & -c \\ c & a \end{pmatrix}$$

Now, $J^2 = -1$ implies a = 0, c = 1.

Remark 3.2 It is sufficient to define J locally. By uniqueness J will glue. Let s, t be as above. We set $J(s) = -t, J(t) = s$. It has all the required properties. From Theorem 2.1 we have a complex structure associated to g, obviously multiplying the metric g by a positive global function f doesn't change the induced complex structure. For a manifold X a conformal structure is a metric g up to multiplication with a positive function.

Proposition 3.2 If X is a Riemann surface with almost complex structure J then there is a unique conformal structure g on X with $g(Js, Jt) = g(s, t)$ for all sections s, t.

Proof: Locally, say around $x \in X$ in a holomorphic coordinate $z = x + iy$, the metric
$gx(dx, dx) = 1 = gx(dy, dy), gx(dx, dy) = 0,$
has the required properties. By partition of unity we get a global metric g with g(J. , J.) = g(. , .).

Given $g\ J.,J.\ = g\ .,.$ computing locally:

$$g\ dx, dx\ = f = g\ dy, dy\ , g\ dx, dy\ = 0, (g(dx, dy) = g(-dy, dx) = -g(dx, dy))$$

with f > 0. Therefore g is unique up to multiplication with a positive function

Associated hermitian form: Suppose X be a Riemann surface and g a metric which is with

$g(J.,J.) = g(.,.)$.

We introduce a hermitian metric

$\boldsymbol{h} : \mathcal{A}^1_X \otimes \mathcal{A}^1_X \longrightarrow \mathcal{A}_X,$

which for $a, b, c, d, s, t \in \mathcal{A}^1_X$ is defined by

$$\boldsymbol{h}(a + ib, c + id) := \boldsymbol{h}(a, c) + \boldsymbol{h}(b, d) + i(\boldsymbol{h}(b, c) - \boldsymbol{h}(a, d)),$$
$$\Rightarrow \boldsymbol{h}(s, t) := g(s, t) + ig(s, Jt).$$

Curvature: Suppose X is a Riemann surface with Riemann metric g compatible with the almost complex structure, and h the associated hermitian metric.

The Gaussian curvature is the function

$$K := \frac{h(dz,dz)}{2} . \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\boldsymbol{log}\left(\boldsymbol{h}(\boldsymbol{dz}, \boldsymbol{dz})\right)\right)$$
$$= \frac{h(dz,dz)}{2}\left(4 \frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}}\right)\left(\boldsymbol{log}\left(\boldsymbol{h}(\boldsymbol{dz}, \boldsymbol{dz})\right)\right) \because \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) = \left(4 \frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}}\right)$$
$$= 2\boldsymbol{h}(\boldsymbol{dz}, \boldsymbol{dz})\left(\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}}\right)\left(\boldsymbol{log}\left(\boldsymbol{h}(\boldsymbol{dz}, \boldsymbol{dz})\right)\right)$$

We are to prove that this function is well-defined. We obtain a global 2–form by setting

$$Vol = \frac{i}{2h(dz, dz)} dz \wedge d\bar{z}$$

In order to see that this is independent of the choice of the coordinate z and therefore defines a global form, let $z' = f(z)$ be a coordinate change (f holomorphic). We get

$$\frac{i}{2h\ dz',dz'}dz' \wedge d\bar{z}' = \frac{i\partial_Z(f)\bar{\partial}_Z(\bar{f})}{2h(\partial_Z\ f\ dz,\partial_Z(f)dz)}dz \wedge d\bar{z} = \frac{i}{2h\ dz,dz}dz \wedge d\bar{z}.$$

The decomposition into $(0,1)$ and $(1,0)$-forms implies a decomposition of the derivative d, $\mathcal{A}_X \xrightarrow{d=(\partial,\bar{\partial})} \mathcal{A}_X^{1,0} \oplus \mathcal{A}_X^{0,1} \xrightarrow{d=\bar{\partial}+\partial} \mathcal{A}_X^2$ With

$\partial(f) = \partial_z(f)dz, \quad \bar{\partial}(f) = \bar{\partial}_z(f)d\bar{z}$,
$\bar{\partial}(fdz) = \bar{\partial}_z(f)d\bar{z} \wedge dz,\ \partial(fd\bar{z}) = \partial_z(f)dz \wedge d\bar{z}$
We get $\partial\bar{\partial}\left(log\left(h(dz,dz)\right)\right) = \partial_z\bar{\partial}_z\ log(h(dz,dz))dz \wedge d\bar{z}$. This defines a global 2-form, because if $z' = f(z)$ then
$\partial\bar{\partial}\left(log\left(h(dz',dz')\right)\right) = \partial\bar{\partial}\ \left(log\left(h(dz,dz)\right) + \log\left(\partial_z(f)\overline{\partial_z(f)}\right)\right).$
and $\partial\bar{\partial}log(\ \partial_z(f)\overline{\partial_z(f)}) = 0.$

The curvature function is obtained by comparing this 2 -form with Vol:

$$\partial\bar{\partial}\left(log\left(h(dz,dz)\right)\right) = -iK \cdot Vol$$

Now, we have the following results

Theorem 3.1 There is unique metric (up to multiplication with a constant) on the upper half plane H that is invariant under the action of $SL_2(\mathbb{R})$.

Theorem 3.2 Suppose X is a Riemann surface, and choose metric g with $g(J\cdot,J\cdot) = g(\cdot,\cdot)$. Show that on the fundamental cover $p : \tilde{X} \to X$ we get an induced metric $\tilde{g}$ with $\tilde{g}(J\cdot,J\cdot) = \tilde{g}(\cdot,\cdot)$, which is invariant under the action of $\pi_1(X)$.

Conformal Structure

Let M be a Riemannian manifold and metrics be in equivalence relation

$$g_1 \sim g_2 \text{ if } g_1 = e^{2w}g_2, w \in C^\infty(M).$$

It is Weyl equivalence, and an equivalence class of metrics is called a conformal structure on M. Diffeomorphisms of M which preserve the conformal structure are called conformal transformations, A Riemannian manifold M with a metric g is called locally conformally flat if every point $p \in M$ has a coordinate neighbourhood U such that

$$g|U = e^{2\omega}\sum_{i=1}^{2n}(dx^i)^2 \tag{4}$$

It turns out that every Riemann surface is locally conformally flat.

Theorem 4.1 The set of conformal classes of metrics and the set of complex structures are in one-to-one canonical correspondence for Riemann surfaces.

Complex structure generated by Metric

Specifying a complex structure completely specifies the conformal structure, and vice-versa. One consequence of this theorem is that the notion of a biholomorphism and a conformal transformation are equivalent.
Let M, g be a 1-Complex dimensional differentiable manifold with a metric g. In local coordinate $x, y : U \subset M \to \mathbb{R}^2$ one has

$$g = adx^2 + 2bdxdy + cdy^2, a > 0, c > 0, ac - b^2 > 0 \tag{5}$$

Definition: Two metrics g and $\bar{g}$ are called conformally equivalent if they differ by a function on M
$$g \sim \bar{g} \Leftrightarrow g = f\bar{g}, f : M \to \mathbb{R}_+ \tag{6}$$

The relation (6) defines the classes of conformally equivalent metrics.

Remark 4.1 The angles between tangent vectors are the same for conformally equivalent metrics.
We show that there is one to one correspondence between the conformal equivalence classes of metrics on an orientable two-manifold M and the complex structure on M. In terms of the complex variable $z = x + iy$ one rewrites the metric as

$$g = Adz^2 + 2Bdzd\bar{z} + \bar{A}d\bar{z}^2, \ A \in \mathbb{C}, B \in \mathbb{R}, B > |A| \tag{7}$$

With

$$a = 2B + A + \bar{A}, \ b = i(A - \bar{A}), c = 2B - A - \bar{A} \tag{8}$$

A coordinate $w : U \to \mathbb{C}$ is called conformal if the metric in this coordinate is of the form

$$g = e^{\varphi} dw d\bar{w} \tag{9}$$

i.e. it is conformally equivalent to the standard metric of $\mathbb{R}^2 = \mathbb{C}$

$$dw d\bar{w} = du^2 + dv^2, \ \ w = u + iv.$$

Remark 4.2 If $F : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ is an immersed surface in $\mathbb{R}^3$ then the first fundamental form (dF, dF) induces a metric on U. When the standard coordinate (x, y) of $\mathbb{R}^2 \supset U$ is conformal, the parameter lines

$$F(x, \Delta m), F(\Delta n, y), x, y \in \mathbb{R}, n, m \in \mathbb{Z} \Delta \to 0$$

Comprise an infinitesimal square net on the surface.

Theorem 4.2 Every compact Riemann surface admits a conformal Riemannian metric.

Proof: Each point $p \in M$ possesses a local parameter $z_p : U_p \to D_p \subset \mathbb{C}$, where D_p is a small open disc. Since M is compact there exists a finite covering $\cup_{i=1}^{n} U_{P_i} = M$. For each i choose a smooth function $m_i : D_{P_i} \to \mathbb{R}$ with

$m_i > 0$ on D_i, $m_i = 0$ on $C \setminus D_i$.

$m_i(zp_i)dzp_i d\bar{z}p_i$ is a conformal metric on U_{P_i}. The sum of these metrics over $i = 1, \dots \dots, n$ yields a conformal metric on M.
Let us show how one finds conformal coordinates. The metric (7) can be written as follows (we suppose $A \neq 0$)

$$g = s(dz + \mu d\bar{z})(d\bar{z} + \bar{\mu} dz), s > 0, \tag{10}$$

Where

$$\mu = \frac{A}{2B}(1 + |\mu|^2), \ s = \frac{2B}{1+|\mu|^2}.$$

Here $|\mu|$ is a solution of the quadratic equation

$$|\mu| + \frac{1}{|\mu|} = \frac{2B}{|A|},$$

which can be chosen $|\mu| < 1$

$$|\mu| = \frac{1}{|A|}\left(B - \sqrt{B^2 - |A|^2}\right). \tag{11}$$

Comparing (10) and (9) we get

$$dw = \lambda(dz + \mu d\bar{z})$$
or
$$dw = \lambda(d\bar{z} + \bar{\mu} dz).$$

In the first case the map $w(z, \bar{z})$ satisfies the equation

$$w_{\bar{z}} = w_z \tag{12}$$

and preserves the orientation $w: U \subset \mathbb{C} \to V \subset \mathbb{C}$ since $|\mu| < 1$: for the map $z \to w$ written in terms of the real coordinates $z = x + iy;\ w = u + iv$ one has

$du \wedge dv = |w_z|^2(1 - |\mu|^2)dx \wedge dy$.

In the second case $w : U \to V$ inverses the orientation

Definition: Equation (12) is called the Beltrami equation and $\mu(z, \bar{z})$ is called the Beltrami coefficient.

Let us postpone for a moment the discussion of the proof of existence of solutions to the Beltrami equation and let us assume that this equation can be solved in a small neighbourhood of any point of M.

Theorem 4.3 Let M be a 1-complex-dimensional orientable manifold with a metric g and an oriented atlas $((x_\alpha, y_\alpha) : U_\alpha \to \mathbb{R}^2)_{\alpha \in A}$ on M. Let $(x; y) : U \subset M \to \mathbb{R}^2$ be one of these coordinate charts with a point $P \in U, z = x + iy, \mu(z, \bar{z})$ the Beltrami coefficient (11) and $w_\beta(z; \bar{z})$ be a solution to the Beltrami equation (12) in a neighbourhood $V_\beta \subset V = z(U)$ with $P \in U_\beta = z^{-1}(V_\beta)$. Then the coordinate w_β is conformal and the atlas $w_\beta: U_\beta \to \mathbb{C}_{\beta \in B}$ defines a complex structure on M.

Proof: To prove the holomorphicity of the transition function let us consider two local parameters $w: U \to \mathbb{C}, \tilde{w}: \tilde{U} \to \mathbb{C}$ with a non-empty intersection $U \cap \tilde{U} \neq \emptyset$. Both coordinates are conformal

$g = e^{\varphi} dw d\bar{w} = e^{\tilde{\varphi}} d\tilde{w} d\bar{\tilde{w}}$,

which happens in one of the two cases

$$\frac{\partial \tilde{w}}{\partial \bar{w}} = 0 \text{ or } \frac{\partial \tilde{w}}{\partial w} = 0 \qquad (13)$$

Only. The transition function $\tilde{w}(w)$ is holomorphic and not antiholomorphic since the map $w \to \tilde{w}$ preserves orientation. Repeating the arguments of the proof of Theorem 4.3 one immediately observes that conformally equivalent metrics generate the same complex structure. Finally, we obtain the following:

Theorem 4.4 Conformal equivalence classes of metrics on an orientable 1-complex-manifold M are in one to one correspondence with the complex structures on M.

On Solution to the Beltrami Equation

For the real-analytic case $\mu \in C^w$ the existence of the solution to the Beltrami equation was known already to Gauss. It can be proven using the Cauchy-Kowalewski theorem.

Theorem 4.5 (Cauchy-Kowalewski)

Let

$$\frac{\partial^m u_i}{\partial x_0^m} = F_i\left(x_0, x, u, \frac{\partial^{m_0 + \cdots + m_n}}{\partial x_0^{m_0} \ldots \partial x_n^{m_n}} u\right)$$

$$i = 1, \ldots, k,\ \ x \in \mathbb{R}^n$$

$$\sum_{j=0}^{n} m_j \leq m, \quad m_0 < m, \quad m \geq 1$$

be a system of k partial differential equations for k functions $u_1(x, x_0), \ldots, u_k(x, x_0)$.

The Cauchy problem

$$\left.\frac{\partial^j u_i}{\partial x_0^j}\right|_\sigma = \emptyset_{ij}(x), \qquad i = 1, \ldots, k;\ j = 0, \ldots, m-1,$$

where $\sigma = \{(x, x_0), x_0 = 0, x \in \Omega_0, \Omega_0 is\ a\ domain\ in\ \mathbb{R}^n\}$ with real-analytic data (all $F_i, \emptyset_{ij}$ are real-analytic functions of all their arguments), has a unique real-analytic solution $u(x, x_0)$ in some domain $\Omega \subset \mathbb{R}^{n+1}$of variables (x, x_0) with $\Omega_0 \subset \Omega$.
In terms of real variables

$$z = x + iy, \quad w = u + iv, \mu = p + iq$$

the Beltrami equation reads as follows:

$$\binom{u}{v}_y = \frac{1}{(1+p)^2+q^2}\begin{pmatrix} 2q & p^2+q^2-1 \\ 1-p^2-q^2 & 2q \end{pmatrix}\binom{u}{v}_x \tag{14}$$

If μ is real-analytic and $|\mu| < 1$ all the coefficients in (14) are real-analytic, which implies the existence of a real-analytic solution to the equation. Solutions to the Beltrami equation exist in much more general case but the proof is much more involved.
Recall that a function is of H $\ddot{o}$lder class of order $\alpha(0 < \alpha < 1)$ on $W, f \in C^\alpha(W)$ if there exists a constant K such that

$$|f(p) - f(q)| \leq K|p - q|^\alpha, \forall\ p, q \in W$$

If all mixed n-th order derivatives of f exist and are C^α then $f \in C^{n+\alpha}(W)$

Theorem 4.6 Let $z : U \to V \subset \mathbb{R}$ be a coordinate chart at some point $p \in U$ and $\mu \in C^\alpha(V)$ be the Beltrami coefficient. There is a solution $w(z, \bar{z})$ to the Beltrami equation of the class $w \in C^{\alpha+1}(W)$ in some neighbourhood W of the point $(p) \in W \subset V$.

Sketch of the proof of Theorem 4.6

The Beltrami equation can be rewritten as an integral equation using

Lemma 4.1 ($\bar{\partial}$-Lemma)

Given $g \in C^\alpha(V)$ the formula

$$f(z) = \frac{1}{2\pi i}\int_V \frac{g(\xi)}{\xi - z} d\xi \wedge d\bar{\xi}$$

defines a $C^{\alpha+1}(V)$ solution to the equation

$$f_{\bar{z}}(z) = g(z).$$

In case $g \in C^\alpha$ or $g \in C^1$ this lemma is a standard result in complex analysis.
The $\bar{\partial}$-Lemma implies that the solution of

$$w(z) = h(z) + \frac{1}{2\pi i}\int_V \frac{\mu(\xi) w_{\xi}(\xi)}{\xi - z} d\xi \wedge d\bar{\xi} \tag{15}$$

where h is holomorph, satisfies the Beltrami equation. The proof of the existence of the solution to the integral equation (15) is standard: it is solved by iterations. Let us rewrite the equation to be solved as

$$w = Tw, \tag{16}$$

where T_w is the right-hand side of (15). Let us suppose that there complete metric space H such that

i) $TH \subset H$
ii) T is a contraction in H, i. e. $\|Tw - Tw'\| < c\|w - w'\|$ for any $w, w' \in H$ with some c < 1.

Then there exists a unique solution $w^* \in H$ of (16) and this solution can be obtained from any starting point $w_0 \in H$ by iteration

$$w^* = \lim_{n\to\infty} T^n w_0. \tag{17}$$

The theorem above holds true also after replacing $\alpha \to \alpha + n, n \in \mathbb{N}$.

Classification of Riemann Surfaces

Mathematicians like to classify everything, up to isomorphism, and Riemann surfaces are no exception. Their classification is given by the *Uniformization theorem*. Before we begin to define the concepts we will need to state the Uniformization theorem let us state a special case;

Theorem 5.1 Uniformization theorem for simply-connected $(\pi_1(\mathcal{M}) = \{e\})$ Riemann surfaces Up to conformal equivalence, there exist three simply-connected Riemann surfaces;

1) $\hat{C} \equiv C \cup \{\infty\}$ the Riemann sphere
2) C the complex plane
3) $\Delta = \{z \in C \mid |z| < 1\}$ the unit disk.

In order to state the Uniformization theorem for arbitrary Riemann surfaces, we must first discuss covering maps, automorphism groups, and freely discontinuous group actions.

Automorphisms: An automorphism of a manifold $\mathcal{M}$ is an biholomorphism $f : M \to M$. The set of automorphisms is denoted $Aut(\mathcal{M})$ and forms a group under composition.

Universal cover: Let $\mathcal{M}$ be any connected Riemann surface. The universal cover of $\mathcal{M}$ is a simply-connected Riemann surface Σ, equipped with a surjective map $\pi : \Sigma \to \mathcal{M}$. In addition, every point $p \in \mathcal{M}$ has a neighbourhood U where $\pi^{-1}(U)$ (the pre-image, not the inverse) is a countable disjoint union of sheets

$$\pi^{-1}(U) = \bigcup_{\alpha \in Z} V_\alpha$$

Each V_α (a sheet) is homeomorphic to U via π.

Covering groups: If $\pi : \Sigma \longrightarrow \mathcal{M}$ is a covering map, then there is a group G of homeomorphisms of Σ such that the quotient space Σ/G is isomorphic to $\mathcal{M}$. G is called the covering group, and is isomorphic to $\pi_1(\mathcal{M})$.

One may also inquire as to when a group *G* of homeomorphisms gives rise to a surface Σ/G with projection map equal to the quotient map. The condition is that of free discontinuous action.

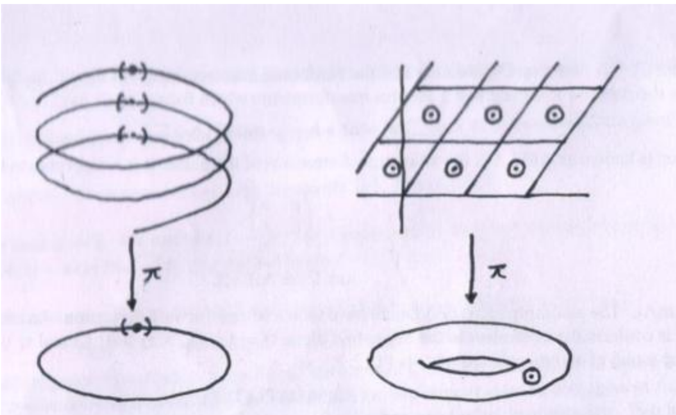

Figure 5.1: A cartoon of two spaces, $\mathbb{S}^1$ and $\mathbb{T}^2$, with their universal covers $\mathbb{R}$ and $\mathbb{C}$ and projection map $\pi : \mathbb{R} \to \mathbb{S}^1$ and $\pi : \mathbb{C} \to \mathbb{T}^2$

A group G acts freely discontinuously at a point $p \in \Sigma$ if $\exists$ a neighbourhood *U* of *p* s.t. $gU \cap U = \phi \ \forall g \in G - \{e\}$.

If the action is freely discontinuous at every point, the quotient space S/*G* is a manifold.
If $\mathcal{M}$ is a Riemann surface and Σ is its universal cover, the complex structure pulls back via the projection map π and the covering group consists of conformal automorphisms; it is a subgroup of Aut (Σ).

Uniformization Theorem for an arbitrary Riemann surfaces

Theorem 5.2 Every Riemann surface $\mathcal{M}$ is conformally equivalent to Σ/G, where

$$\Sigma = \begin{Bmatrix} \hat{\mathbb{C}} \\ \mathbb{C} \\ \Delta \end{Bmatrix}$$

is the universal cover of $\mathcal{M}$, and G is a subgroup of Aut (Σ) admitting a freely discontinuous action on M . In addition, $\mathrm{G} \simeq \pi_1(\mathcal{M})$

Automorphisms: Subgroups of the group of automorphisms of the three Riemann surfaces play an important role in Theorem 5.2., so it is a good idea to know what *Aut* is for each surface.

Aut $\hat{\mathbb{C}}$**:** We use

$\hat{\mathbb{C}} \simeq \mathbb{CP}^1 \equiv \mathbb{P}^1$

with the isomorphism given in homogeneous coordinates on $\mathbb{P}^1$ by

$$([z_1, z_2]) \mapsto \frac{z_1}{z_2},\ \ ([1,0]) \equiv \{\infty\} \tag{18}$$

The action of $GL(2,\mathbb{C})$ on $\mathbb{C}^2$ projects to an action of $PL(2,\mathbb{C}) \equiv \{GL(2,\mathbb{C})/\lambda, \lambda \in \mathbb{C}^*\}$ on $\mathbb{P}^1$. Then, $PL(2,\mathbb{C})$ is the group Aut $(\hat{\mathbb{C}})$ whose action on $\hat{\mathbb{C}}$ is

$$\begin{pmatrix} z_1' \\ z_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \mapsto z' = \frac{az+b}{cz+d}, \tag{19}$$

by virtue of equation (18). Such a transformation (the RHS of (19)) is called a M ö bius transformation.

$Aut\,(\hat{\mathbb{C}}) \simeq PL(2,\mathbb{C})$.

Aut $(\mathbb{C})$**:** The conformal automorphisms of $\mathbb{C}$ will be those automorphisms of $\hat{\mathbb{C}}$ which fix the point ∞. Clearly a M ö bius transformation which fixes ∞ must have $c = 0$;

$z' = az + b,\ \ a \in \mathbb{C}^*, b \in \mathbb{C}$

This group is known as Aff $(1,\mathbb{C})$, the affine transformations of the plane. It is isomorphic to the group of matricies of the form

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$$

Thus,

$Aut\,(\mathbb{C}) \simeq Aff\,(1,\mathbb{C})$.

Aut (Δ): The automorphisms of Δ do not have such a straightforward derivation. An overview is presented: The unit disk is conformally equivalent to the upper-half plane $\mathcal{U} = \{z \in \mathbb{C} | \mathcal{J}(z) > 0\}$ (Δ and $\mathcal{U}$ will be used interchangeably).

The group of automorphisms of $\mathcal{U}$ is $PL(2,\mathbb{R})$, so

$Aut\,(\Delta) \simeq PL(2,\mathbb{R})$.

Theorem 5.3 With three classes of exceptions, the fundamental group π_1 of a Riemann surface is non-abelian.

- The only surfaces up to conformal equivalence with trivial fundamental group are $\mathbb{C}, \hat{\mathbb{C}}$ and Δ.
- The only surfaces (u.t.c.e.) with $\pi_1 \simeq \mathbb{Z}$ are $\mathbb{C}^*, \Delta^*$, and $\Delta_r = \{z \in \mathbb{C} | r < |z| < 1\}$.
- The only surfaces (u.t.c.e.) with $\pi_1 \simeq Z \oplus Z$ are tori $\mathbb{C}/\Lambda(\omega,\eta)$, Λ a lattice in $\mathbb{C}$.

These conformal classes of surfaces are known as exceptional.

Module of Riemann surfaces

Two Riemann surfaces can have the same underlying topological space, and yet be conformally-inequivalent. The set of conformally-inequivalent surfaces over the same topological space is known as a moduli space. Is there structure on moduli spaces? The answer to this question is yes, generically, and the structure can be very interesting. The general solution is complicated, so we will be content with examining a few important examples.

To aid our study, we shall make use of the following theorem.

Theorem 6.1 Two Riemann surfaces have the same universal cover S and covering groups conjugate in Aut (Σ) if and only if they are conformally equivalent. That is, for $G_1, G_2 \subset \mathrm{Aut}(\Sigma)$, there exists $g \in \mathrm{Aut}(\Sigma)$ s.t. $gG_1g^{-1} = G_2$.

Surfaces with universal cover $\acute{\mathbb{C}}$: As Aut $(\acute{\mathbb{C}})$ is the Lie group PL $(2, \mathbb{C})$. If $\mathbb{C} = \acute{\mathbb{C}} \cup \infty$, then Aut $(\acute{\mathbb{C}})$ is a M öbius transformation, from (2).

Proposition 6.1 The only Riemann surface with universal cover $\acute{\mathbb{C}}$ is $\acute{\mathbb{C}}$ itself.

Proof: First we show that M öbius transformations fix at least one point of $\acute{\mathbb{C}}$.
Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PL(2, \mathbb{C})$. Then, a fixed point satisfies $z = \frac{az+b}{cz+d}$, which simplifies to

$$z = \frac{a-d}{2c} \pm \frac{1}{2c}\sqrt{(d-a)^2 - 4cb} \quad c \neq 0$$
$$z = \frac{b}{d-a} \quad c = 0, a \neq d$$
$$z = \infty \quad c = 0, a = d, b \neq 0$$

These equations clearly have solutions for any element of $PL(2, \mathbb{C})$. Thus, we see that every element of Aut $(\acute{\mathbb{C}})$ fixes at least one point of $\acute{\mathbb{C}}$, and since no proper subgroup of PL $(2, \mathbb{C})$ can act freely discontinuously on $\acute{\mathbb{C}}$, we have the desired result.

An obvious consequence of this proposition is that the moduli space of genus zero surfaces is a one-point set. In fact, all three of the simply-connected surfaces have one-point moduli spaces.

Surfaces with universal cover $\mathbb{C}$: Recall that Aut $(\mathbb{C})$ = Aff $(1, \mathbb{C})$, with $z \mapsto az + b$. We will make use of the fact that

Theorem 6.2 If the universal covering space of $\mathcal{M}$ is $\mathbb{C}$, then $\mathcal{M}$ is conformally equivalent to $\mathbb{C}, \acute{\mathbb{C}}$, or T^2, a torus.

The respective covering groups are $\{e\}$, $\mathbb{Z}$ and $\mathbb{Z} \oplus \mathbb{Z}$.
Examine first the case $G = \mathbb{Z}$. We can take $z \mapsto z + 1$ as a generator, so that the covering map $\pi: \mathbb{C} \to \mathbb{C}^*$ is just the exponential map $\pi(z) = \exp(2\pi\sqrt{-1}z)$. The moduli space for $\mathbb{C}^*$ is again trivial.

$G = \mathbb{Z} \oplus \mathbb{Z}$ is a bit more complicated. We shall make use of Theorem 6.1. Consider a lattice in $\mathbb{C}$;
$\Lambda(\omega, \eta) = \{m\omega + n\eta \mid m, n \in \mathbb{Z}; w, \eta \in \mathbb{C}^* \text{ linearly independent}\}$.

Clearly it is a discrete group, isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$, and the quotient $\mathbb{C}/\Lambda(\omega, \eta)$ is a torus (**see figure 5.1**). When two lattices do are give conformally equivalent tori? By Theorem 6.1, the two lattices must be conjugate in Aut(C). Before we establish conditions for the conjugacy of two lattices, we elucidate one further complication. Two lattices $\Lambda(\omega_1, \eta_1)$ and $\Lambda(\omega_2, \eta_2)$ may be identical; that is, they are composed of the same set of points in C.

Lemma 6.1 The two lattices define the same set of points if the two basis elements of one lattice are linear combinations of the other two (with integer coefficients);

$$\begin{pmatrix} \omega_1' \\ \eta_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_1 \\ \eta_2 \end{pmatrix}$$

with $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $GL(2, \mathbb{Z})$. Furthermore, multiplying the entire matrix by any integer will yield the same lattice; we want $PL(2, \mathbb{Z})$.

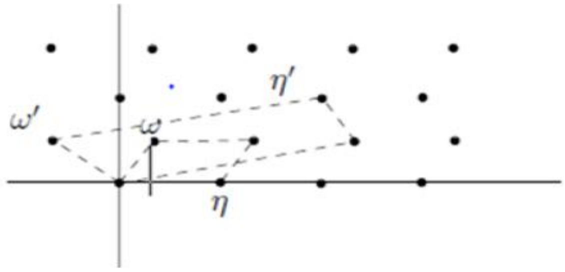

Figure 6.1 Two bases defining identical lattices in $\mathbb{C}$

Proposition 6.2 The conjugacy class of $\Lambda(\omega, \eta)$ in Aut $(\mathbb{C})$ is the set of lattices of the form $\Lambda(a\omega, a\eta)$, with $a \in \mathbb{C}^*$.

Proof: An element $(a, b) \in \mathrm{Aut}(\mathbb{C})$ acts on a generator $h_\omega : z \mapsto z + \omega$ of the covering group as $(a, b) \cdot h_\omega(z) = a(z + \omega) + b,$ and thus

$$(a, b)h_\omega(a, b)^{-1}(z) = z + a\omega.$$

Now, we define $\tau = \eta/\omega$ and without loss of generality, choose $\Im(\tau) > 0$ (i.e. since any two points in the lattice define it, choose them appropriately: see figure 6.1). Furthermore, we choose $a = \frac{1}{\omega}$, so that every lattice is conjugate to one of the form $\Lambda(1, \tau)$. Combining this result and that of Lemma 6.1, we see that two lattices are equivalent if they are related by the Mobius transformation

$$\tau' = \frac{a\tau + b}{c\tau + d} \text{with} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PL(2, \mathbb{Z}).$$

Theorem 6.3 The moduli space of the torus is $\mathcal{U}/PL(2, \mathbb{Z})$.

Surfaces with universal cover $\mathcal{U}$: Except for the four surfaces mentioned previously $(\hat{\mathbb{C}}, \mathbb{C}, \mathbb{C}^*, \text{and } T^2)$, all Riemann surfaces have $\mathcal{U}$ as their universal covering space. There are many subgroups of $\mathrm{Aut}(\mathcal{U})$ which act freely discontinuously, the fixed-point-free Fuchsian groups. Rather than beginning a long discourse, we shall leave them for another time. For now, we will find the moduli spaces of the rest of the exceptional Riemann surfaces $\Delta, \Delta^*, \text{and } \Delta_r$.

Surface	π_1	Moduli Space
$\hat{\mathbb{C}}$	$\{e\}$	$\{pt\}$
$\mathbb{C}$ $\mathbb{C}^*$ T^2	$\{e\}$ [illegible] $\mathbb{Z}\oplus\mathbb{Z}$	$\{pt\}$ $\{pt\}$ $\mathcal{U}/PL(2,\mathbb{Z})$
Δ, Δ^* Δ_r	$\{e\}$ [illegible] [illegible]	$\{pt\}$ $\{pt\}$ $1, \infty$

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