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RESEARCH ARTICLE

GENERALIZATION OF FIXED POINT THEROMS AND SELFMAP OF F-ORBITALLY COMPLETE D-METRIC SPACE

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ABSTRACT

In this paper, some fixed point theorems are proved in D-metric spaces. The Generalization of fixed point theorems and selfmaps in F-orbitally compete D-metric space which include unique fixed point results in Dolhare (2016), Dhage *et al.* (2003), Dolhare (?), Dolhare and Bele (2016), generalized fixed point therems in F-orbitally complete D-metric spaces as a special cases.

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1. INTRODUCTION

The fixed point theory is an important and major topic of the nonlinear functional analysis and finds some nice applications to other branches of mathematics. For proving the theorems on selfmaps and some nonlinear problems that arise in various physical and biological process.The various topological spaces such as metric, Banach, Hilbert, locally convex and Housdorff spaces etc are used in formulating the fixed point theorems. The famous French Mathematician H. Poincare (1854-1912) first recognized the importance of study of the nonlinear problems and predicated that the future Mathematics will mainly deal only with non-linearity and since then several methods have been developed and employed in the study of nonlinear equations.The fixed point technique is one of the most important and powerful tools that has been generally used for solving self maps and fixed point theorems which is major core part of the nonlinear functional analysis.It is known that the D-metric ϱ is a continuous function in the topology of D-Metric convergence. The generalization of a D-metric up to n variables appear in Dhage and Dolhare (2003). Also Das and Gupta (1975), Edelstien (1962), Poincare (1886), Rakotch (1962), Gu Seman (1970) etc. Worked on Fixed point theory and proved fixed fixed point theorems. We need the following fixed point theorem for selfmaps proved in Dolhare and Dhage (2003) and Rhoades (1977), Caccippoli (1930) in the sequel K.Iseki proved the following theorem for selfmap in complete metric space. It is well known that the notion of metric function is generalization of the idea of the distance function.

2 : Fixed Point and Contraction

Definition 2.1 :Dolhare U.P(2): Let X be a set and $f: X \to X$ be a map. A point $x \in X$ is called a fixed point of f if $f(x) = x$. i.e. If f is defined on the real number by $f(x) = x^2 - 7x + 12$. We know that $x = 3,4$ are roots of the equation. Let us consider $f(x) = x$. Where $f(x) = \dfrac{x^2+12}{7}$ then $x = 3$ and $x = 4$ are two fixed points of $f(x)$.

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Definition 2.2 :Dolhae U.P(1) Let (X,d) be a Metric Space. A mapping $f: X \to X$ is called a contraction or contraction mapping if there exists a number $\alpha < 1$ such that

$$d(f(x), f(y)) \leq \alpha\, d(x,y) \qquad \ldots\ldots\ldots (2.1)$$

Where x , y $\neq 0$ For all $x, y \in X$ and α is called the contraction coefficient.

Thus, a contraction maps points closer together in particular, for every $x \in X$, and any $r > 0$ all points y in the ball $B_r(x)$ are map into a ball $B_s(f_x)$ with $s < r$. Some times a map satisfying equation (2.1) with $\alpha = 1$ is also called a contraction, and then map satisfying (2.1) with $\alpha < 1$ is called a strict contraction. It follows from (2.1) that a contraction mapping is uniformly continuous. If $f: X \to X$, then a point $x \in X$ such that

$$f(x) = x \qquad \ldots\ldots\ldots (2.2)$$

is called a fixed point of f.

The contraction mapping theorem states that a strict contraction on a complete metric space has a unique fixed point. The contraction mapping theorem is only one example of fixed point theorems. There are fixed point theorems for maps satisfying (2.1) with $\alpha = 1$, the Schauder fixed point theorem states that a continuous mapping on a convex compact subset of Banach space has a fixed point.

Definition 2.3 : (A Meir and Emmett Keeler) (5) Let (X, ρ) be a complete Metric space and f be a mapping of X it to itself. If there exists an $\alpha < 1$ and for all $x, y \in X$ such that

$$\rho(f(x), f(y)) \leq \alpha \cdot \rho(x,y), \qquad \alpha < 1$$

Then $f(\xi) = \xi$ then f has unique fixed point.

Dolhare U.P. generalized the following theorem for complete metric space as follows.

Theorem 2.1: Dolhare U.P.(1) *Let (X,d) be a complete metric space and $T: X \to X$ be satisfying*

$$(T_x, T_y) \leq \alpha \cdot d(x,y), \quad \forall\, x, y \in X \qquad \ldots\ldots\ldots (2.3)$$

Where $0 \leq \alpha < 1$. Then T has a unique fixed point in X.

Theorem 2.2 : **Dolhare U.P.and Bele C.D.(3)** *Let f be a self maps of f-orbitally complete D- metric space X satisfying* $(x,y,z)\rho(C, f_y, f_z) = \lambda\, \rho$ *(where $0 \leq \lambda \leq 1$ then f has a unique fixed point.*

Theorem 2.3 : Dolhare U.P.(2) :*Let (X ,ρ) be a complete metric space and f be a self map on X such that f^2 is continuous if $g: f(x) \to X$ such that g f(x) ⊂ $f^2(x)$ and g (f(x))= f (g (x)) both sides are defined for all x,y ϵ f(x). Then f and g have unique common fixed point.*

Most important results in this direction has been obtained by Presic (1965) by generalizing the Banach contraction mapping principle.

Theorem 2.4: (S.B. Presic (17)) Let $X, d)$ be a complete metric space, k is a positive integer and $T: X^k \to X$ a mapping satisfying the following contractive type condition

$$d\big(T(x_1, x_2, \ldots x_k), T(x_2, x_3, \ldots x_{k+1})\big) \leq q_1 d(x_1, x_2) + q_2 d(x_2, x_3) + \ \ldots\ldots + q_k d(x_k, x_{k+1}) \qquad \ldots\ldots\ldots (2.4)$$

For every $x_1, x_2, x_3, \ldots x_k, x_{k+1}$ where $q_1, q_2, q_3, \ldots q_k$ are nonnegative constants such that $q_1 + q_2 + q_3 + \cdots + q_k < 1$ then there exists a unique point x in X such that

$$T(x, x, \ldots\ldots, x) = x$$

Theorem 2.5 Let (*X*,*d*) be a generalized metric space , and the mapping

$F : X \to X$ is uniformly locally contractive which satisfies (2.3). Then *F* has a Unique fixed point.

Theorem 2.6 : If f is a selfmap of a complete metric space (X, d), f is onto, and there is a constant a >1 such that, d(f_x, f_y) ≥ ∝d (*x*,*y*) for all $x,y \in Y$ then f has a unique fixed point.

T.Taniguchi generalized this theorem of K.Iseki by considering the following conditions

$d(f_x, f_y) > \min\{d(x,y), d(x,f_x), d(y,f_y)\}$ for all $x,y \in X$ and $x \neq y$. ......(2.5)

He proved that each continuous self map f of impact metric space satisfying the above condition has a fixed point. Khan (1968) used the square root condition for h < 1 to obtain a common fixed point of two underlining continuous self mapping.

$d(S_x, T_y) \leq h\{d(X,S_x).d(y,T_y)\}^{1/2}$ ......(2.6)

for all Rhoades(19) proved the following theorem for self map.

Theorem 2.7 : Let f be a self map of complete metric space X satisfying for some positive integer P,

$d(f^p{}_x, f^p{}_y) \leq k\, d(x,y)$ ..... (2.7)

for all x ,y ∈ X, where $0 \leq \lambda < 1$. Then f has a unique fixed point $u \in X$ and f is continuous at u..

Definition 2.4:- $T : X \to X$ is called $(\in, ^a)$ Uniformly locally contractive if it is locally contractive at all points of $x \in X$ and $\in, ^a$ do not depend on x i.e.

$$d(x,y) < \in \;\Rightarrow\; d(Tx,Ty) <^a d(x,y) \text{ for all } x,y \in X$$

Further the fixed point theorem is generalized by Kanan (1968) in a g.m.s. stated as follows.

Theorem 2.8 (Kanan) (15) Let X be a Complete metric Space and $F : X \to X$ is a mapping such that,

$$d(Fx,Fy) \leq \quad [d(x,Fx) + d(y,Fy)] \text{ for all } x, y \in X \qquad ………(2.8)$$

Where $\in\ ]0,1[$ Then F has a Unique fixed point.

Theorem 2.9 : Dolhare (2) Let $f: X \to X$ be a contraction of the complete metric space (X,ρ) so that $d(f(x),f(y)) \leq \lambda\, d(x,y)$ for some $0 \leq \lambda \leq 1$, and let x_0 be any point X. Then the sequence $\{x_n\}$ defined by $x_{n+1} = f(x_n)$ converges to unique fixed point x. Further more for any n^{th} value we have

$$d(x_n, x) \leq \frac{\lambda^n}{1-\lambda}\, d(x_0, f(x_0))$$

Then f has unique fixed point.

Theorem 2.10 : Let (X,ρ) be a complete metric space suppose f is contraction mapping and λ is a constant for f and λ^n is the constant for f^n. Then f^n is also a contraction and f^n has a fixed point.

Proof : Firstly we will show that the theorem is true for $n = 2$. Since f is a contraction consider $\lambda < 1$, then

$$d(f(x), f(y)) \leq \lambda\, d(x,y)$$

We can apply f to $f(x)$ and $f(y)$ such that

$$d(f^2(x), f^2(y)) \leq \lambda\, d(f(x), f(y))$$

Since $d(f(x), f(y)) \leq \lambda\, d(x,y)$

$$d(f^2(x), f^2(y)) \leq \lambda\, d(f(x), f(y)) \leq \lambda^2 d(x,y)$$

Thus, $\quad d(f^2(x), f^2(y)) \leq \lambda^2 d(x,y)$

Since $\lambda < 1, \lambda^2 < 1$ then f^n is a contraction,

Since f^n is a contraction then it is true that f^{n+1} is also contraction

$d\left(f^{n+1}(x), f^{n+1}(y)\right) \leq \lambda^{n+1} d\left(f(x), f(y)\right) \leq \lambda^{n+1} d(x,y)$

Thus, $\quad d\left(f^{n+1}(x), f^{n+1}(y)\right) \leq \lambda^{n+1} d(x,y)$

Thus by induction the theorem is true for all n. If $f(x) = x$, then

$f^2(x) = f\left(f(x)\right) = f(x) = x$

Then $\quad f^n(x) = x$

Hence by induction f^n is contraction and f^n has the unique fixed point.

Theorem 2.11 : Let (X, ρ) be a complete metric space suppose f is contraction mapping and λ is a constant for f and λ^n is the constant for f^n. Then f^n is also a contraction and f^n has a fixed point.

Proof : Firstly we will show that the theorem is true for $n = 2$. Since f is a contraction consider $\lambda < 1$, then

$d\left(f(x), f(y)\right) \leq \lambda\, d(x,y)$

We can apply f to $f(x)$ and $f(y)$ such that

$d\left(f^2(x), f^2(y)\right) \leq \lambda\, d\left(f(x), f(y)\right)$

Since $d\left(f(x), f(y)\right) \leq \lambda\, d(x,y)$

$d\left(f^2(x), f^2(y)\right) \leq \lambda\, d\left(f(x), f(y)\right) \leq \lambda^2 d(x,y)$

Thus, $\quad d\left(f^2(x), f^2(y)\right) \leq \lambda^2 d(x,y)$

Since $\lambda < 1, \lambda^2 < 1$ then f^n is a contraction,

Since f^n is a contraction then it is true that f^{n+1} is also contraction

$d\left(f^{n+1}(x), f^{n+1}(y)\right) \leq \lambda^{n+1} d\left(f(x), f(y)\right) \leq \lambda^{n+1} d(x,y)$

Thus, $\quad d\left(f^{n+1}(x), f^{n+1}(y)\right) \leq \lambda^{n+1} d(x,y)$

Thus by induction the theorem is true for all n. If $f(x) = x$, then

$f^2(x) = f\left(f(x)\right) = f(x) = x$

Then $\quad f^n(x) = x$

Hence by induction f^n is contraction and f^n has the unique fixed point

In our Result we show that the existence of a fixed point for a Kanan contraction in orbitally complete Generalized metric space is actually a generalization of ………
further natural generalizations of metric space are recently made in Dolhare and Dhage(4) under the title ‘‘ D-Metric function’’.

3 D-Metric Space

In the present paper we prove a fixed point theorem similar to theorem 2.7 which is its generalization in D-metric spaces.

The concept of a D-metric spaces Introduced by Dolhare and Dhage (2003 is as follows.

A non empty set X together with a function ϱ: X x X x X → (0 , ∞) is called a D-Metric space with D-Metric ϱ, denoted by (X, ϱ)

if ρ satisfies the following properties :

i. $\varrho(x,y,z) = 0 \Leftrightarrow x=y=z$ (coincidence)
ii. $\varrho(x,y,z) = \varrho(\varrho\{x,y,z\})$ (symetry)
iii. $\varrho(x,y,z) \leq \varrho(x,y,a) + \varrho(x,a,z) + \varrho(a,y,z)$ (Tetrahedral inequality)

It is known that D-metric ϱ is a continuous function in the topology of D-metric convergence the generalization of a D-metic upto n variables appear in Dhage and Dolhare (2003). By definition of D-Metric space, theorem (2.7) in metric space we generalized in to self map of a f-orbitally complete D-Metric space as follows.

4 F-Orbit ally complete Mapping

Definition 4.1 (4) :- Let T be a mapping of generalized Metric Space (g.m.s) (X,d) into itself is called as T-Orbitally complete iff any Cauchy Sequence

$\{x_n\} \subseteq \{x, Tx, T^2x, T^3x, ...\}$ for $x \in X$ converges in X itself

By taking concept of F-orbitally Complete maps then a slight generalization of Theorem 2.8 is as follows.

Theorem 4.2 (8) If (X,d) be a generalized metric space and $F : X \to X$ is a mapping such that,

$$d(Fx, Fy) \leq \quad [d(x, Fx) + d(y, Fy)] \quad \text{for all } x, y \in X$$

Where $\in \;]0,1[$. If X is F-orbitally complete. Then F has a Unique fixed point in X.

Dolhare U.P. generalized theorem 2.7 in D-Metric Space as follows.

Theorem 4.3 : Let f be a selfmap of a f-orbitally complete D-Metric space X satisfying for some positive integer p, such that
$d(f^p{}_x, f^p{}_y, f^p{}_x) \leq \lambda\ \varrho(x,y,z)$

for all x, y, z ∈ X, where $0 \leq \lambda < 1$ the f has a unique common fixed point u ∈ X and f is f-orbitally continuous at u.
In the following section are proved the main result of this paper

5. Main results

Let f: X → X then by an orbit of f at a point x ∈ X is a set $O_f(x)$ is X defined by

$O_f(x) = \{ x, f_x, f^2{}_x \_\ \_\ \_ \}$ ....(5.1)

D-Metric spaces is said to be *f*-orbitally complete if every D-Cauchy sequence $\{X_n\} \subseteq O_f(x)$, x ∈ X converges to a point in X. Again a D-Metric space X is said to be *f*-orbitally bounded if $\delta(O_f(x)) < \infty$ for each x ∈ X. Finally a mapping f is called *f*-orbitally continuous if $\{x_n\} \subset O_f(x)$. x ∈ X, $x_n \to x^*$, implies $fx_n \to fx^*$.

Theorem 5 .1 : Let f be a self map of a *f*-orbitally complete D-metric space X satisfying

$(f^n{}_x, f^n{}_y, f^n{}_z) \leq a_n\, d\,(x,y,z)$ ....(5.2)

For all x,y,z ε X where $a_n \geq 0$, n ∈ N

Further if $\sum_{n=1}^{\infty} an < \infty$ then ....(5.3)

f has a unique fixed point u∈X for large n, f_n is continuous if f is f-orbitally continuous at u

Proof. :Since $a_n \geq 0$, n ∈ N and $\sum_{n=1}^{\infty} an < \infty$, we have lim $a_n = 0$

Therefore there exists a positive integer p such that $a_p < ½$ and

$\varrho\,(f^px, f^py, f^pz) \leq a_p\ \varrho(x,y,z) \leq ½\, \varrho(x,y,z)$ ....(5.4)

for all x,y,z ∈ X. Now an application of theorem 4.3 yields that f has a unique common fixed point u ∈ X and f is f-orbitally continuous at u and the inequality shows that f^p satisfies a Lipschitz type condition on X w.r.t a D-Metric ϱ and so by a theorem in (4), then f^p is continuous on X and in particular at u this completes the proof. We note that the following form of the well-known D-Contraction mapping principle proved in Dhage and Dolhare (4) follows from our theorem (5.1) as a corollary.

Corollary 2.1 :Let f be a slelfmap of a f-orbitally complete D-metric space X satisfying

$\varrho\,(f_x, f_y, f_z) \leq \lambda\ \varrho(x,y,z)$ ....(5.5)

for all x,y,z ∈ X, where $0 \leq \lambda < 1$ then f has a unique fixed point

Proof :By repeated application of the inequality (5.5) we get ,

$\varrho(f^n{}_x,f^n{}_y,f^n{}_z) \leq \lambda^n \varrho(x,y,z)$ For all x,y,z ∈ X. As $0 \leq \lambda < 1$.

$\sum_{n=1}^{\infty} \lambda^n < \infty$, and so the desired conclusion follows by an application of theorem 5.1

Example 5.1 Define $a_n = 1/n^p$, $p > 1$ then (5.4) is satisfied and so the conclusion of theorem 5.1 holds with a_n. In this case f is nonexpansive and hence it is continuous on X which is proved in Dolhare (4).A slight generalization of theorem 2.7 and Theorem 4.3 is as follows.

Therorem 5.2 : Let f be a selfmap of complete Metric space X satisfying

$(f^n{}_x,f^n{}_y,f^n{}_z) \leq a_n\varrho(x,y,z)$ for all x,y,z ∈ X where $a_n \geq 0$, n ∈ N. Further if there exists a positve integer m such that lim inf $a_n < 1/a_m$ then f has a unique fixed point u ∈ X and f is continuous at u.

Proof: If a_m<1, then f^m is a D-Contraction and by theorem (4.3), f has a unique fixed point u ∈ X & f is f-orbitally continuous at u.Suppose that $a_m \geq 0$ By hypothesis . $\lim_n inf\ an < \frac{1}{am} < 1$.

Then there is a positive integer p such that $a_n < 1/a_m < 1$ so that f^m become a D-Contraction mapping on X. So by theorem 4.3 f has a unique fixed point and f is orbitally continuous at u.

Conclusion

In the present paper we used contraction mapping for to find unique fixed point of self maps in complete Metric Space.

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