Available online at http://www.journalcra.com

International Journal of Current Research
Vol. 9, Issue, 10, pp.59199-59204, October, 2017

INTERNATIONAL JOURNAL OF CURRENT RESEARCH

ISSN: 0975-833X

RESEARCH ARTICLE

A NOTE ON UNIVARIATE VARIABILITY OF FUZZY MEASURES

Rajan. D and *Beulah. A

Department of Mathematics, T.B.M.L College, Poraiyar -609307, Tamilnadu, India

ARTICLE INFO

Article History:
Received 28th July, 2017
Received in revised form
19th August, 2017
Accepted 21st September, 2017
Published online 31st October, 2017

Key words:

Fuzzy sets orders, Fuzzy Measures, Fuzzy Measures Functions, α - cuts of Fuzzy measures and order wealth.

ABSTRACT

In this paper, we study Fuzzy set orders that compare the variability or the dispersion of Fuzzy Measures. The most important and common order that has studied in this paper are the fuzzy set order and the dispersive order. We also study in this paper the excess wealth order which is found to be useful in the increasing order of application. Various related order are also examined in the paper.

Citation: Rajan. D and Beulah. A 2017. "A note on univariate variability of fuzzy measures", *International Journal of Current Research*, 9, (10), 59199-59204.

1. INTRODUCTION

This paper contemplates on Fuzzy set orders that compare the variability or the dispersion of Fuzzy Measures. Following the Workshop Chang, *et al.* (Bhattacharjee, 2003) a study has been made on the variability and dispersive behavior of Fuzzy Measures. Ahmadi *et al.* (Kwakernaak, 1978) has attempted sum univariate set orders on record values. Bassal *et al.* (Ahamadi and Arghami, 2001) has studied in detail variability orders and mean differences. Belzunce (Ou Jinping and Wang Guangyuan, 1989) has studied extensively on the characterization of right spread order by the increasing set order. Belzunce *et al.* (Puri and Ralescu, 1986) have studied dispersive ordering and characterizations of ageing classes. Bhattacharjee *et al.* (Zhang yue, 1990) has attempted some generalized variables ordering among life distribution with reliability applications. Bhattacharjee (Bassan *et al.*, 1999) has dealt in detail discrete set, Characterization, equivalence and applications. Battacharjee *et al.* (Belzunce, 1999) have made an attempt on set equivalence are convex order distribution and applications. Boustsikas *et al.* (Belzunce *et al.*, 1996) have studied on the distance between set order fuzzy Measures. Chan *et al.* (Bhattacharjee, 1991) have extensively studied set ordering among functions with a application to reliability. In the light of these journals we apply the notion to the case of Fuzzy Measures on the dispersive nature of Fuzzy Measures.

****Corresponding author:*** **Beulah, A.**
Department of Mathematics, T.B.M.L College, Poraiyar -609307, Tamilnadu, India.

2. Preliminaries

We introduce in this section some notions of fuzzy sets and operations of fuzzy numbers. Kwakernack (Boutsikas and Vaggelatou, 2002) introduce the concept of a fuzzy measures as a function $F: X \rightarrow F(R)$ where (X, A, μ) is a measurable space and $F(R)$ denotes all piecewise continuous functions $U: R\rightarrow [0, 1]$. Puri and Ralescu [1986] defined the notion of a Fuzzy Measures as a function $F : X\rightarrow F(R^n)$, where (X, A, μ) is a measurable space, and $F(R^n)$ denotes all functions $U : R^n \rightarrow [0, 1]$ such that $\{x \in R^n, U(x) \geq \alpha\}$ is nonempty and compact for each $\alpha \in [0, 1]$. We introduce a notion of a Fuzzy Measures (Bhattacharjee, 2003) slightly different than that of Kwakernakk (Boutsikas and Vaggelatou, 2002) and Puri (Chan *et al.*, 1990). We define it as a measurable fuzzy set valued function $x : X \rightarrow F_0(R)$, where R is the real line, (X, A, μ) is a probability space, $F_0(R) = \{A : R\rightarrow [0, 1]\}$ and $\{x\in R; A(x) \geq \alpha\}$ is a bounded closed interval for each $\alpha \in (0, 1)$. Let U be a nonempty usual set, P(U) denote the set of all subsets in U and F(U) denote the set of all fuzzy subsets in U, and F(U) denote the set of all fuzzy subsets in U. For $A \in F(U)$ we define two subsets of U as follows:

$A_\alpha = \{x\in U; A(x) \geq\alpha\}$ for any $\alpha\in [0, 1]$,
$A_\alpha = \{x \in U; A(x) >\alpha\}$ for any $\alpha\in [0, 1]$,

Where A(x) is the membership functions of A. These are known as α-cuts of the fuzzy set A. Without loss of generality in the sequel X_α, F_α, G_α, denote the respective α-cut functions.
$A_\alpha = [A^-_\alpha, A^+_\alpha]$

Where A^-_α= inf A_α, A^+_α = sup A_α

Definition 2:1

Let X be an universal set and A be a fuzzy subset of X with membership function A(x)∈ [0,1]. The α- level set of the fuzzy set A is defined as

A_α = {x ; A(x) ≥α } where A_0 is the closure of the set {x ; A(x) ≠0}

Definition 2:2

The fuzzy set A is called a normal fuzzy set if there exists x such that A(x) = 1.

A is called a convex fuzzy set if A (λx +(1- λ) y ≥ min { A(x) , A(y) } for λ∈ [0,1]

Definition 2:3

Let (X, A, μ) be a measurable space λ and a set valued mapping.

X → I(R) = {[x,y] ; x,y ϵ R, x ≤ y},
w ↔ λ (w) = [λ^- (w), λ^+(w)]
Then λ (w) = [λ^- (w), λ^+(w)] is called a set interval if λ^-(w) and λ^+(w) are both measures on (X, A,μ).

Definition 2:4

Let (X, A, μ) be a measurable space. A fuzzy set valued mapping.

x : X → F_0(R) is called a fuzzy random variable if for every B ϵ *B* and every x ϵ (0,1),
x_α^{-1}(B) = {w ϵ X; x_α(w) ∩ B ≠ ϕ}ϵ A.
x_α(w) = [x^-_α(w), x_α^+(w)] is a set valued interval for every α ϵ (0, 1) and
x(w) = $\bigcup_{\alpha\epsilon(0,1)} \alpha\, x_\alpha(w)$
= $\bigcup_{\alpha\epsilon(0,1)} \alpha\, [x_\alpha^- w, X_\alpha^+(w)]$

3. Variability ordering of fuzzy measures:

Definition: 3:1

Let X and Y be two fuzzy sets such that

$$\Pi\left[\bigcup_{\alpha\epsilon(0,1)} \alpha\ \phi(x_\alpha)\right] \le \Pi\left[\bigcup_{\alpha\epsilon(0,1)} \alpha\ \phi(y_\alpha)\right] \qquad (3.1)$$

for all set functions ϕ : R→R, provided the bounded exists.

Then X is said to be small than Y in the set order denoted as $X_\alpha \le Y_\alpha$ set functions are functions that take on the larger values over region of the from $(-\infty, a) \cup (b, \infty)$, for a < b, therefore if (3.1) holds then y is more likely to take on extreme values than X_α i.e. Y is more valuable than Y_α. It functions ϕ_1 *and* ϕ_2 defined by $\phi_1(X_\alpha) = x, \phi_2(X_\alpha) = -x$ both set from (3.1) it is easily follows that $X_\alpha \le Y_\alpha$ implies

$$\Pi\left[\bigcup_{\alpha\in(0,1)} X_\alpha(x)\right] = \Pi\left[\bigcup_{\alpha\in(0,1)} Y_\alpha(y)\right] \qquad (3.2)$$

provided bounded exists.

It is useful to note that if

$$\Pi\left[\bigcup_{\alpha\in(0,1)} \alpha\, X_\alpha\ (x)\right] = \Pi\left[\bigcup_{\alpha\in(0,1)} \alpha\, Y_\alpha\ (y)\right]$$

Then

$$\int_{-\infty}^{\infty} [\alpha F_\alpha(U_\alpha) - \alpha G_\alpha(U_\alpha)] dU_\alpha = \int_{-\infty}^{\infty} \alpha \bar{F}(U_\alpha) - \alpha \bar{G}(U_\alpha) dU_\alpha = 0 \qquad (3.3)$$

A function ϕ defined on an open interval (a, b) is said to be convex or compact it for each x, yϵ (a, b) and each λ, 0 ≤ λ ≤ 1 we have ϕ [λ (x) + (1 – λ)y] ≤ λ ϕ(x) + (1 – λ) ϕ(y) provides the integral exists. Where $\bar{F}$ and $\bar{G}$ are measurable functions and F and G are continuous function of x and y respectively.

The function ϕ defined on ϕ(x) is convex or compact.

∴ from (3.1) and (3.2) it follows that $X_\alpha \le Y_\alpha$ implies

μ (X) ≤ μ (Y) (3.4)

Whenever μ(Y) is less than ∞

For a fixed a the function ϕ(a) defined by $\phi_a(x) = (x-a)_+$, and the function ϕ_a defined by $\phi_a(x) = (a-x)_+$ are both compact therefore, if $x \le y$, then

$$\Pi\left[\cup_{\alpha\in(0,1)} \alpha\ (X_\alpha - a)_+\right] \le \Pi\left[\cup_{\alpha\in(0,1)} \alpha\ (Y_\alpha - a)_+\right] \text{ For all a} \qquad (3.5)$$

And $$\Pi\left[\left[\cup_{\alpha\in(0,1)} \alpha\ (a - X_\alpha)_+\right]\right] \le \Pi\left[\cup_{\alpha\in(0,1)} \alpha\ (a - Y_\alpha)_+\right] \qquad (3.6)$$

for all a provided the bounded exists.

Alternatively, using a simple integration by parts, it is seen that (3.5) and (3.6) can be rewritten as

$$\int_{\bar{x}}^{\infty} \alpha\, \overline{F_\alpha}(U_\alpha) dU_\alpha \le \int_x^{\infty} \alpha\, \overline{G_\alpha}(U_\alpha) dU_\alpha \qquad for\ all\ x \qquad (3.7)$$

and

$$\int_{-\infty}^{x} \alpha\, F_\alpha(U_\alpha) dU_\alpha \le \int_{-\infty}^{x} \alpha\, G_\alpha(U_\alpha) dU_\alpha \qquad for\ all\ x \qquad (3.8)$$

Provided the integrals exists.
Which completes the proof.

Theorem: 3:1

Let X and Y be two fuzzy sets such that $\Pi(X) = \Pi(Y)$ then

(i)$X \le Y$ if and only if

$$\int_x^{\infty} \bar{F}(u) du \le \int_x^{\infty} \bar{G}(u) du \quad for\ all\ x \qquad (3.9)$$

(ii)$X \le Y$ if and only if

(iii) $\int_{-\infty}^{x} F(u)du \leq \int_{-\infty}^{x} G(u)du \ for\ all\ x$ (3.10)

Proof:

By adding a to both sides of inequality ϕ

$\Pi\left[\cup_{\alpha\in(0,1)} \alpha \ (X_\alpha - a)_+\right] + a \leq \Pi\left[\cup_{\alpha\in(0,1)} \alpha \ (Y_\alpha - a)_+\right] + a$ for all a,

it is seen that ϕ can be written as

$Max\left[\cup_{\alpha\in(0,1)} \alpha \ (X_\alpha, a)\right] \leq \Pi\left[Max\left[\cup_{\alpha\in(0,1)} \alpha \ (Y_\alpha, a)\right]\right]$ for all a

then when $\Pi\left[\cup_{\alpha\in(0,1)} \alpha \ X_\alpha\right] = \Pi\left[\cup_{\alpha\in(0,1)} \alpha \, Y_\alpha \right]$ (3.11)

then (3.9) is equivalent to

$\cup_{\alpha\in(0,1)} \alpha \ X_\alpha \leq \cup_{\alpha\in(0,1)} \alpha \ Y_\alpha \quad \Rightarrow \quad X \leq Y.$

In a similar manner (3.6) can be rewritten.

Which completes the proof.

Theorem: 3:2

Let X and Y be two fuzzy sets such that $\boldsymbol{\Pi}(X) = \boldsymbol{\Pi}(Y)$ then $\boldsymbol{X \leq Y}$ if and only if

$\Pi[X_\alpha - a] \leq \Pi[Y_\alpha - a]$ for all aϵR, for αϵ(0,1). (3.12)

Proof:

Clearly, if $\left[\cup_{\alpha\in(0,1)} \alpha \ X_\alpha\right] \leq \left[\cup_{\alpha\in(0,1)} \alpha \ Y_\alpha\right]$ then so suppose that $\Pi\left[\cup_{\alpha\in(0,1)} \alpha \ (X_\alpha - a)\right] \leq \Pi\left[\cup_{\alpha\in(0,1)} \alpha \ (Y_\alpha - a)\right]$ for all a ϵ R holds. without loss of generality it can be assumed that

$\Pi\left[\cup_{\alpha\in(0,1)} \alpha \ X_\alpha\right] \leq \Pi\left[\cup_{\alpha\in(0,1)} \alpha \ Y_\alpha\right] = 0$ then

$$\Pi\left[\bigcup_{\alpha\in(0,1)} \alpha \ (X_\alpha - a)\right] = a + 2\int_{a}^{\infty} \alpha F_\alpha(u_\alpha)du_\alpha$$

$$= -a + 2\int_{-\infty}^{a} \alpha F_\alpha(u_\alpha)du_\alpha \quad (3.13)$$

then the results follows from (3.9) and (3.10).

Theorem: 3:3

Let X and Y be two fuzzy sets with measurable functions F and G, respectively and with equal finite means. Then each of the following two statements is a necessary and sufficient condition for $X \leq Y$:

$$\int_{0}^{\mu} F_\alpha^{-1}(u)du_\alpha \geq \int_{0}^{\mu} G_\alpha^{-1}(u)du_\alpha$$

For all $\mu\epsilon[0,1]$ (3.14) and

$$\int_{\mu}^{1} F_\alpha^{-1}(u)du_\alpha \leq \int_{\mu}^{1} G_\alpha^{-1}(u)du_\alpha$$

For all $\mu\epsilon[0,1]$ (3.15) for all αϵ(0,1).

Proof:

Since, $\Pi\left[\cup_{\alpha\in(0,1)} \alpha \ X_\alpha\right] = \int_0^1 \cup_{\alpha\in(0,1)} \alpha \ F_\alpha^{-1}(u_\alpha)du_\alpha$

and $\Pi\left[\cup_{\alpha\in(0,1)} \alpha \ Y_\alpha\right] = \int_0^1 \cup_{\alpha\in(0,1)} \alpha \ G_\alpha^{-1}(u_\alpha)du_\alpha$

and since

$$\Pi\left[\bigcup_{\alpha\in(0,1)} \alpha \ X_\alpha\right] = \Pi\left[\bigcup_{\alpha\in(0,1)} \alpha \ Y_\alpha\right]$$

It follows that for any μϵ [0,1],

the inequality

$$\int_{\mu}^{1} \bigcup_{\alpha\in(0,1)} \alpha \ F_\alpha^{-1}(u_\alpha)du_\alpha \leq \int_{\mu}^{1} \bigcup_{\alpha\in(0,1)} \alpha \ G_\alpha^{-1}(u_\alpha)du_\alpha \quad 3.16$$

Thus, it is equivalent to the inequality.

$$\int_{0}^{\mu} \bigcup_{\alpha\in(0,1)} \alpha \ F_\alpha^{-1}(u_\alpha)du_\alpha \geq \int_{0}^{\mu} \bigcup_{\alpha\in(0,1)} \alpha \ G_\alpha^{-1}(u_\alpha)du_\alpha \quad 3.17$$

It follows that (3.14) & (3.15) are equivalent.

Thus, we just need to show that $\cup_{\alpha\in(0,1)} \alpha \ X_\alpha \leq \cup_{\alpha\in(0,1)} \alpha \ Y_\alpha \Rightarrow X \leq Y$ is equivalent to 3.14. We only give the proof for the case when the measurable functions F_α and G_α of X_α and Y_α are continuous the proof for the general case is similar, through notationally more complex. Without loss of generality, suppose that F_α and G_α are not identical. Since $\Pi X_\alpha = \Pi Y_\alpha$ it follows that F_x and G_α must cross each other at least once. If it is either (3.7) or (3.16) hold, then if there is a first time that F_α crosses G_α it must cross it there from below. Similarly, if there is a last time that F_α crosses G_α, it also must cross it there from below. (Thus, it there is a finite number of crossings, then it must be odd). Let (y_0, μ_0), (y_1, μ_1) and (y_2, μ_2) be three consecutive crossing points. Note that (y_0, μ_0) may be $(-\infty, 0)$.(we then adopt the convention that $0.(-\infty) \equiv 0)$ and that (y_2, μ_2) may be $(\infty, 1)$ (we then adopt the convention that $0.(\infty) \equiv 0)$. Note that by the continuity assumption we have $(\mu_i = F_\alpha(y_i) = G_\alpha(y_i))$, $i = 0,1,2$. Assume that $\cup_{\alpha\in(0,1)} \alpha \ X_\alpha \leq_{cx} \cup_{\alpha\in(0,1)} \alpha \ Y_\alpha$ then

$$\int_{y_2}^{\infty} \bigcup_{\alpha\in(0,1)} \alpha \ \bar{F}_\alpha(x)dx \leq \int_{y_2}^{\infty} \bigcup_{\alpha\in(0,1)} \alpha \ \bar{G}_\alpha(x)dx \quad (3.16)$$

Thus $\int_{\mu_2}^{1} \cup_{\alpha\in(0,1)} \alpha \ F_\alpha^{-1}(u_\alpha)du_\alpha = y_2(1-\mu_2) + \int_{y_2}^{\infty} \cup_{\alpha\in(0,1)} \alpha \ \bar{F}_\alpha(x)dx$

$\leq y_2(1-\mu_2) + \int_{y_2}^{\infty} \cup_{\alpha\in(0,1)} \alpha \ \bar{G}_\alpha(x)dx$ (by equation 3.16)

$= \int_{\mu_2}^{1} \cup_{\alpha\in(0,1)} \alpha \ G_\alpha^{-1}(u_\alpha)du_\alpha$ (3.17)

Now, for U_α ϵ $[\mu_1, \mu_2]$ we have that

$\bigcup_{\alpha\in(0,1)}\alpha\ F_{\alpha}^{-1}(u_{\alpha})-\bigcup_{\alpha\in(0,1)}\alpha\ G_{\alpha}^{-1}(u_{\alpha})\ \leq 0$.

Thus $\int_{\mu}^{1}(\bigcup_{\alpha\in(0,1)}\alpha\ F_{\alpha}^{-1}(u_{\alpha})-\bigcup_{\alpha\in(0,1)}\alpha\ G_{\alpha}^{-1}(u_{\alpha}))du_{\alpha}$ is increasing in $\mu\in[\mu_1,\mu_2]$.

Therefore form (3.17), we get that

$\int_{\mu}^{1}(\bigcup_{\alpha\in(0,1)}\alpha\ F_{\alpha}^{-1}(u_{\alpha})du_{\alpha}\leq\int_{\mu}^{1}(\bigcup_{\alpha\in(0,1)}\alpha\ G_{\alpha}^{-1}(u_{\alpha})du_{\alpha}$ For $\mu\in[\mu_1,\mu_2]$ (3.18).

From $\bigcup_{\alpha\in(0,1)}\alpha\ X_{\alpha}\leq\bigcup_{\alpha\in(0,1)}\alpha\ Y_{\alpha}$

We also have

$$\int_{-\infty}^{y_0}\bigcup_{\alpha\in(0,1)}\alpha\ F_{\alpha}(x)dx\ \leq\ \int_{-\infty}^{y_0}\bigcup_{\alpha\in(0,1)}\alpha\ G_{\alpha}(x)dx \qquad (3.19)$$

Thus,

$$\int_{0}^{p_0}\bigcup_{\alpha\in(0,1)}\alpha\ F_{\alpha}^{-1}(u_{\alpha})du_{\alpha}=y_0\mu_0-\int_{-\infty}^{y_0}\bigcup_{\alpha\in(0,1)}\alpha\ F_{\alpha}(x)dx\geq y_0\mu_0-\int_{-\infty}^{y_0}\bigcup_{\alpha\in(0,1)}\alpha\ G_{\alpha}(x)dx \qquad (3.20)$$

By equation (3.19)

$=\int_{0}^{\mu_0}\bigcup_{\alpha\in(0,1)}\ G_{\alpha}^{-1}(u_{\alpha})du_{\alpha}$.

Now, for $u_\alpha\in[\mu_0,\mu_1]$

we have that $\bigcup_{\alpha\in(0,1)}\alpha\ F_{\alpha}^{-1}(u_{\alpha})-\bigcup_{\alpha\in(0,1)}\alpha\ G_{\alpha}^{-1}(u_{\alpha})\ \geq 0$. Thus $\int_{0}^{\mu}(\alpha\,F_{\alpha}^{-1}(u_{\alpha})-\alpha\,G_{\alpha}^{-1}(u_{\alpha}))du_{\alpha}$ is increasing in $\mu\in[\mu_0,\mu_1]$.

Therefore, from (3.20), we get that

$\int_{0}^{\mu_0}\bigcup_{\alpha\in(0,1)}\alpha\ F_{\alpha}^{-1}(u_{\alpha})du_{\alpha}\geq\int_{0}^{\mu}\ _{\alpha\in(0,1)}^{\cup}\alpha G_{\alpha}^{-1}(u_{\alpha})du_{\alpha}$ for $\mu\in[\mu_0,\mu_1]$. (3.21)

Thus we see from (3.17) and (3.20) that for each $\mu\in[\mu_0,\mu_1]$. either (3.13) or (3.14) holds. Therefore ((3.14) or equivalently (3.15)) holds.

Conversely, assume that ((3.14) or equivalently (3.15)) holds. Then

$$\int_{\mu_2}^{1}\bigcup_{\alpha\in(0,1)}\alpha\ F_{\alpha}^{-1}(u_{\alpha})du_{\alpha}\leq\int_{\mu_2}^{1}\bigcup_{\alpha\in(0,1)}\alpha\ G_{\alpha}^{-1}(u_{\alpha})du_{\alpha} \qquad (3.22)$$

Thus

$$\int_{y_2}^{\infty}\bigcup_{\alpha\in(0,1)}\alpha\ \overline{F_{\alpha}}(x)dx=\int_{\mu_2}^{1}\bigcup_{\alpha\in(0,1)}\alpha\ F_{\alpha}^{-1}(u_{\alpha})du_{\alpha}-y_2(1-\mu_2)$$

$$\leq\int_{\mu_2}^{1}\bigcup_{\alpha\in(0,1)}\alpha\ G_{\alpha}^{-1}(u_{\alpha})du_{\alpha}-y_2(1-\mu_2)\qquad (3.22)\to(3.23)$$

$$=\int_{y_2}^{\infty}\bigcup_{\alpha\in(0,1)}\alpha\ \overline{G_{\alpha}}(x)dx$$

Now, for $x\in[y_1,y_2]$ we have that

$$\bigcup_{\alpha\in(0,1)}\alpha\ \overline{F_{\alpha}}(x)-\bigcup_{\alpha\in(0,1)}\alpha\ \overline{G_{\alpha}}(x)\ \leq\ 0$$

Thus $\int_{y}^{\infty}[\bigcup_{\alpha\in(0,1)}\alpha\ \overline{F_{\alpha}}(x)-\bigcup_{\alpha\in(0,1)}\alpha\ \overline{G_{\alpha}}(x)]dx$ is

increasing in $y\in[y_1,y_2]$. Therefore, from (3.23) we get

$$\int_{y}^{\infty}\bigcup_{\alpha\in(0,1)}\alpha\ \overline{F_{\alpha}}(x)dx\ \leq\int_{y}^{\infty}\bigcup_{\alpha\in(0,1)}\alpha\ \overline{G_{\alpha}}(x)dx\ \ for\ y\in[y_1,y_2] \qquad (3.24)$$

From (14) we also have

$$\int_{0}^{\mu_0}\bigcup_{\alpha\in(0,1)}\alpha\ F_{\alpha}^{-1}(u_{\alpha})du_{\alpha}\geq\int_{0}^{\mu_0}\bigcup_{\alpha\in(0,1)}\alpha\ G_{\alpha}^{-1}(u_{\alpha})du_{\alpha} \qquad (3.25)$$

Thus

$$\int_{-\infty}^{y_0}\bigcup_{\alpha\in(0,1)}\alpha\ F_{\alpha}(x)dx=y_0\mu_0-\int_{0}^{\mu_0}\bigcup_{\alpha\in(0,1)}\alpha\ F_{\alpha}^{-1}(u_{\alpha})du_{\alpha}$$

$\leq y_0\mu_0-\int_{0}^{\mu_0}\bigcup_{\alpha\in(0,1)}\alpha\ G_{\alpha}^{-1}(u_{\alpha})du_{\alpha}$ by (3.25)

$$=\int_{-\infty}^{y_0}\bigcup_{\alpha\in(0,1)}\alpha\ G_{\alpha}(x)dx \qquad (3.26)$$

Now, for $x\in[y_0,y_1]$ we have that

$$\bigcup_{\alpha\in(0,1)}\alpha\ F_{\alpha}(x)-\bigcup_{\alpha\in(0,1)}\alpha\ G_{\alpha}(x)\leq 0$$

Thus

$\int_{-\infty}^{y}\left(\bigcup_{\alpha\in(0,1)}\alpha\ F_{\alpha}(x)-\bigcup_{\alpha\in(0,1)}\alpha\ G_{\alpha}(x)\right)dx$ is decreasing in $\boldsymbol{y}\in[\boldsymbol{y_0},\boldsymbol{y_1}]$. Therefore, from (3.26) we get that $\int_{-\infty}^{y}\bigcup_{\alpha\in(0,1)}\alpha\ F_{\alpha}(x)dx\ \leq\int_{-\infty}^{y}\bigcup_{\alpha\in(0,1)}\alpha\ G_{\alpha}(x)dx$ for $\boldsymbol{y}\in[\boldsymbol{y_0},\boldsymbol{y_1}]$ (3.27)

Thus we see from (3.24) and (3.27) that for each y ϵ R either (3.7) or (3. 8) hold. Therefore

$$\bigcup_{\alpha\in(0,1)}\alpha\ X_{\alpha}\leq\bigcup_{\alpha\in(0,1)}\alpha\ Y_{\alpha}$$

Which completes the proof.

Theorem: 3:4

Let X and Y be independent fuzzy sets. Then $\boldsymbol{X}\leq\boldsymbol{Y}$ if and only if $\Pi[\phi(X_{\alpha},Y_{\alpha})]\leq\Pi[\phi(Y_{\alpha},X_{\alpha})]$ for all $\phi\in G$ for $\alpha\in[0,1]$. (3.28)

Proof:

Suppose that $\Pi[\phi(X_{\alpha},Y_{\alpha})]\leq\Pi[\phi(Y_{\alpha},X_{\alpha})]$ holds.

Let ψ be a univariate set function.

Define $\phi(x_\alpha, y_\alpha) = \psi(x_\alpha)$ then $\phi \in G_{cx}$ and from (3.28) we see that $\boldsymbol{X}_\alpha \leq_{cx} \boldsymbol{Y}_\alpha$.

Conversely, suppose that $\boldsymbol{X}_\alpha \leq \boldsymbol{Y}_\alpha$ let $\phi \in G$ and let Y_α be another fuzzy measure, independent if X_α and Y_α such that $Y_\alpha =_{st} Y_\alpha$

define ψ by $\psi(\boldsymbol{x}_\alpha) \equiv \boldsymbol{\Pi}[\emptyset(X_\alpha, Y_\alpha)\text{-}\emptyset(Y_\alpha, X_\alpha)]$.

Thus the independence of X and $\bar{Y}$ it follows that ψ is compact. Therefore, since $X_\alpha \leq_{xc} Y_\alpha$ it follows that

$\Pi[\emptyset(X_\alpha, Y_\alpha)] - \Pi[\emptyset(Y_\alpha, X_\alpha)] = \Pi[\psi(X_\alpha)] \leq \Pi[\psi(Y_\alpha)] = 0.$

Theorem: 3:5

Let X_α and Y_α be two fuzzy sets with measurable functions F_α and G_α respectively and with finite integral. Then $X_\alpha \leq Y_\alpha$ if and only if

$$\frac{1}{1-P}\int_P^1 [F_\alpha^{-1}(u) - G_\alpha^{-1}(u)]du_\alpha \leq \int_P^1 [F_\alpha^{-1}(u) - G_\alpha^{-1}(u)]du_\alpha$$

For $\alpha \in (0,1)$, forall $P \in [0,1]$

Proof:

Denote $\Delta = \Pi X_\alpha - \Pi Y_\alpha$

Then the set inequality $X_\alpha \leq Y_\alpha$ can be rewritten as $X_\alpha - \Delta \leq Y_\alpha$.

Denote by $F_\alpha\Delta$ the measurable function of $X_\alpha - \Delta$ and note that from theorem (3.3)

we have that $X_\alpha - \Delta \leq Y_\alpha$ if and only if,

$$\int_0^1 F_{\Delta_\alpha}^{-1}(u)du_\alpha \leq \int_P^1 G_\alpha^{-1}(u)du_\alpha \; for\; P \in [0,1] \qquad (3.30)$$

since $F_{\Delta\alpha}(x_\alpha) = F_\alpha(x_\alpha + \Delta)$ for all $x_\alpha \in R$ it follows that $F_{\Delta\alpha}^{-1}(u) = F_\alpha^{-1}(u) - \Delta$ for all $U \in [0,1]$.

Therefore (3.30) is equivalent to

$$\int_P^1 [F_\alpha^{-1}(u) - \Delta]\, du_\alpha \leq \int_P^1 G_\alpha^{-1}(u)\, du_\alpha \quad for\; all\; P \in [0,1];$$

That is, $\int_P^1 [F_\alpha^{-1}(u) - G_\alpha^{-1}(u)]du \leq \int_P^1 [\Pi X_\alpha - \Pi Y_\alpha]\, du_\alpha$ for $\alpha \in (0,1)$ for all $P \in [0,1]$; that is

$\frac{1}{1-P}\int_P^1 [F_\alpha^{-1}(u) - G_\alpha^{-1}(u)]du_\alpha \leq \Pi x_\alpha - \Pi y_\alpha \; for\; all\; P \in (0,1)$ (3.31).

Now, since $\Pi X_\alpha = \int_0^1 F_\alpha^{-1}(u)du_\alpha and\; \Pi Y_\alpha = \int_0^1 G_\alpha^{-1}(u)du_\alpha$ it is seen that (3.31) is equivalent to (3.32).

Closure and Other Properties:

Using equations (3.1) to (3.13) it is easy to prove each of the closure results in the first two parts of the following theorem.

Theorem: 4:1

(a) Let X and Y be two fuzzy sets, then $X_\alpha \leq Y_\alpha \Leftrightarrow -X_\alpha \leq -Y_\alpha$.
(b) Let X_α, Y_α and λ be fuzzy sets such that $[X_\alpha|\lambda = \square] \leq [Y_\alpha]\lambda = \square]$ for all θ in the support of λ. Then $X_\alpha \leq Y_\alpha$ that is the convex order is closed under mixtures.
(c) Let $\{X_{\alpha j}, j = 1,2,\dots\}$ and $\{Y_{\alpha j}, j = 1,2,\dots\}$ be two sequences of fuzzy measures such that $X_{\alpha j} \to X\; and\; Y_{\alpha j} \to y_\alpha\; as\; j \to \infty$. Assume that $\Pi|X_{\alpha j}| \to \Pi|x_\alpha|\; and\; \Pi|Y_{\alpha j}| \to \Pi|y_\alpha|\; as\; j \to \infty$ (3.33).
if $X_{\alpha j} \leq Y_{\alpha j}, j = 1,2,\dots$, then $X_\alpha \leq Y_\alpha$.
(d) Let $X_{\alpha 1}, X_{\alpha 2}, \dots, X_{\alpha m}$ be a set independent fuzzy measures and let $Y_{\alpha 1}, Y_{\alpha 2}, \dots, Y_{\alpha m}$ be another set of independent fuzzy measures . If $X_{i\alpha} \leq Y_{i\alpha}$ for $i = 1,2,\dots,m$, then $\sum_{j=1}^m X_{\alpha j} \leq \sum_{j=1}^m Y_{\alpha j}$ that is convex order is closed under convolutions.

Proof

In order to prove that (c) of theorem (4.1) we will use the characterization of the convex order given theorem (4.2) without loss of generality it can be assumed that $\Pi X_{\alpha j} = \Pi Y_{\alpha i} = \Pi X_\alpha = \Pi Y_\alpha = 0$ for all j. Using the result , Let X_α and Y_α be two fuzzy non-negative measurable space with finite means, then,

$X_\alpha \leq y_\alpha \;\Leftrightarrow\; L(X_\alpha) \leq L(Y_\alpha)$

We have that, $\Pi|X_{j\alpha} - a| = -a + 2\int_{-\infty}^a F_{j\alpha}(u)du$ for all a.

Where F_j denotes the measurable function of $X_{j\alpha}$. In particular, When a = 0,

It is seen that $\Pi|X_{j\alpha}| = 2\int_{-\infty}^0 F_{j\alpha}(u)du_\alpha$.

Therefore $\Pi|X_{j\alpha} - a| = \Pi|X_{j\alpha} - a| + 2\int_0^a F_j(u)du$.

It is seen that, as j→ ∞, the latter expression converges to $\Pi|X_\alpha - a| + 2\int_0^a F_\alpha(u)du_\alpha = \Pi|X_\alpha - a|$,

Where F_α is the measurable function of X_α. That is, for all a, $\Pi|X_{j\alpha} - a| \to \Pi|X_\alpha - a|$, as j→ ∞.

Similarly, $\Pi|Y_{j\alpha} - a| \to \Pi|Y_\alpha - a|$, as j→ ∞.

The Result now follows from theorem (3.2) one way of proving part (d) of theorem (4.1) is the following. Note that (b) of theorem (4.1) can be rephrased as follows. Let Z_1, Z_2 and λ be independent fuzzy measures and Let g_α be a bivariate function such that,

$g_\alpha(Z_1, \lambda) \leq g_\alpha(Z_2, \lambda)$ for all θ in the support of Φ (3.34)

Then $g_\alpha(Z_1, \Phi) \leq g_\alpha(Z_2, \Phi)$ If Z_1 and Z_2 satisfy $Z_1 \leq Z_2$, then the function g, defined by $g_\alpha(Z, \lambda) = Z + \lambda$, Satisfies (3.34). Since the order ≤ is closed under shifts.

Thus we have shown that if $Z_1 \leq Z_2$ and Φ is any fuzzy measure independent of Z_1 and Z_2 then, $Z_1 + \lambda \leq Z_2 + \lambda$ (3.35)

Repeated applications of (3.35) yield part (d) of therefore (4.1)
It should be pointed out, in contrast to part (a) of therefore (2.1), that if X_α and Y_α are such that $X_\alpha \leq Y_\alpha$, it is not necessarily true that $X_\alpha \leq -Y_\alpha$ also, even when

$\Pi X_\alpha = \Pi Y_\alpha = 0$.

This can be seen easily from (4.1) without condition.

REFERENCES

Ahamadi, J, Arghami, N.R. 2001. Some Univariate orders on record values. Communications in Statistics theory and methods, 30, 69-74.

Bassan, B., Denuit, M.Scarsini, M. 1999. Variablity orders and mean differences. Statistics and probability letters, 45, 121-130.

Belzunce, F, Gandel, J.Ruiz, J.M. 1996. Dispersive ordering and characterizations of ageing classes. Statistics and probability letters 28, 321-327.

Belzunce, F. 1999. On a characterization of right spread order by the increasing conve order. Statistics and probability letters, 45,103-110.

Bhattacharjee, M.C. 1991. Some generalized variability orderings among life distributions with reliability applications. *Formal of Applied Probability,* 28, 374-383.

Bhattacharjee, M.C. 2003. Discrete convex ordered lifetimes: characterization, equivalence and applications. Sankh 65, 292-306.

Boutsikas, M.V. and Vaggelatou, E. 2002. On the distance between sets-ordered variables, with applications. Advances in applied probability 34, 349-374.

Chan, W., Proschan, F, Sethuraman, In Block, H.W., Sampson, A.R., Savits, T.H(ed) Topics in statistical Dependence. IMS Lec-ture notes-Monograph series 16, Hayward, California, 121-134 (1990).

Chang, C-S, Chao, X.L., Pinedo, M-Shanthikumar, 1991. IEEE Transactions on Automatic control 36, 1347-1355.

Gu W. and J. Zhou, 1995. A new resolution theory of fuzzy Sets. *J.Northeast Normal. Univ*., 2, 6-7.

Kruse, R. 1982. On the Construction of fuzzy measures, FSS 8:323-327.

Kwakernaak, H. 1979. Fuzzy random variables I, II, Infrom Sci., 15:1-29, 17:153-178.

Liu Dsosu and cheng shaozhong, 1983. Fuzzy Random measure , FSS 11: 135-149.

Ou jinping and wang guangyuan, fuzzy random variables and their probability characters, J.Harbin Architectural and civil engineering institute (1) (1989) 1-10 (in chinese).

Puri M.L.and D. A. Ralescu, 1986. Fuzzy random variables, *J.Math Anal Appl*., 114: 409-422.

Zhang yue, fuzzy random predicting earth quake intensity, in: fu Zizhi *et al*. 1990. Eds., proceedings of the international conference on structural engineering and computation (Beijing Univ. press, Beijing) 695-700.

Zhang, W. 1988. Set-valued Measures and Random Sets. Xi'an jiaotong University Press.

****