ISSN: 0975-833X

Available online at http://www.journalcra.com

International Journal of Current Research
Vol. 11, Issue, 07, pp.5695-5699, July, 2019

DOI: https://doi.org/10.24941/ijcr.35200.07.2019

INTERNATIONAL JOURNAL OF CURRENT RESEARCH

RESEARCH ARTICLE

WHEN ZERO-DIVISOR GRAPH OF SOME SPECIAL IDEALIZATION RINGS ARE DIVISOR GRAPHS

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ARTICLE INFO

Article History:
Received 16th April, 2019
Received in revised form
17th May, 2019
Accepted 20th June, 2019
Published online 31st July, 2019

Key Words:

The idealization rings R,
Divisor graph, Zero-divisor graph.

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ABSTRACT

Let R be a ring with unity and let M be an R-module. Let $R(+)M$ be the idealization of the ring R by the R-module M. In this article, we investigate when $\Gamma(Z\alpha(+)M)$ and $\Gamma(Z_{p^{\alpha_1}}\alpha(+)Z_{p^{\alpha_2}})$ are divisor graphs.

***Citation: Manal Al-Labadi, Wasim Audeh and Anwar Alboustanji,* 2019.** "When zero-divisor graph of some special idealization rings are divisor graphs", *International Journal of Current Research*, 11, (07), 5695-5699.

INTRODUCTION

In this article, all rings are a commutative principal ideal ring with unity. Let **S** be a nonempty set of positive integers and let G_S be the graph whose vertices are the element of S. Two distinct vertices a, b are adjacent if and only if $a|b$ or $b|a$. A graph G is called a divisor graph if there is a set of positive integers S such that $G \cong G_S$. For $\mathbf{S} = \{1,2,\dots,n\}$, the length of longest path in G_S is studied in 9,11,12. In a directed graph G, a vertex is called a receiver if its out-degree is zero and its in-degree is positive. A transmitter is a vertex having positive out-d egree and zero in-degree. A vertex t with a positive in-degree and a positive out-degree is transitive if whenever $u \to t$ and $t \to v$ are edges in G, then $t \to v$ is an edge in G. In 8, divisor graph are investigated. Some results are listed below:

1. No divisor graph contains an induced odd cycle of length 5 or more (proposition 2.1).
2. An induced subgraph of a divisor graph is a divisor graph (proposition 2.2).
3. Complete graphs and bipartite graphs are divisor graphs (proposition 2.5 and Theorem, 2.7).
4. A graph G is a divisor graph if and only if there is an orientation D of G in which every vertex is transmitter, receiver, or transitive (Theorem 3.1).

Divisor graphs are also studied in 1,2.

Another concept of interest in recent years in the concept of zero divisor graph which was introduced by I. Beck in [7] then studied by D. D. Anderson and M. Naseer in [4] in the context of coloring. The definition of zero-divisor graphs in its present form was given by Anderson and Livingston in 5, Theorem 2.3.

A zero-divisor graph of a commutative ring R is the graph $\Gamma(R)$ whose vertices are the nonzero zero-divisors of R, with r and s adjacent if $r \neq s$ and $rs = 0$. In [5], Anderson and Livingston proved that the graph $\Gamma(R)$ is connected with diameter at most 3. The zero divisor graph of a commutative ring has been studied extensively by several authors 3,6,.

For each R, let $Z(R)$ be the set of all zero-divisors of R and $Reg(R) = R\ Z(R)$.

Let M be an R- module. Consider $R(+)M = \{(a,\ m)\colon a \quad R(+)M\}$ and let $(a,\ m)$ and $(b,\ n)$ be two elements of $R(+)M$. Define $(a,\ m) + (b,n) = \{(a+b,\ m+n)\}$ and $(a,m)(b,n) = (ab,\ an+bm)$. Under this definition $R(+)M$ becomes a commutative ring with unity. Call this ring the idealization ring of M in R. For more details, one can look in [10].

Let G be a graph with the vertex set $V(G)$. The degree of a vertex v in a graph G is the number of edges incident with v. The degree of a vertex v is denoted by $\deg(v)$. The complete graph of order n is denoted by K_n, is a graph with n vertices in which any two distinct vertices are adjacent. A star graph is a graph with a vertex adjacent to all other vertices and has no other edges. Recall that a graph G is connected if there is a path between every two distinct vertices. For every pair of distinct vertices x and y of G, let $d(x,y)$ be the length of the shortest path from x to y and if there is no such a path we define $d(x,y) = \infty$. The diameter of G, $diam(G)$, is the supremum of the set $\{d(x,y)\colon x$ and y are distinct vertices of $G\}$.

In our investigation, we start with the following result.

The following result (Figure 1) which was given in 8 as example of a graph which is not divisor graph.

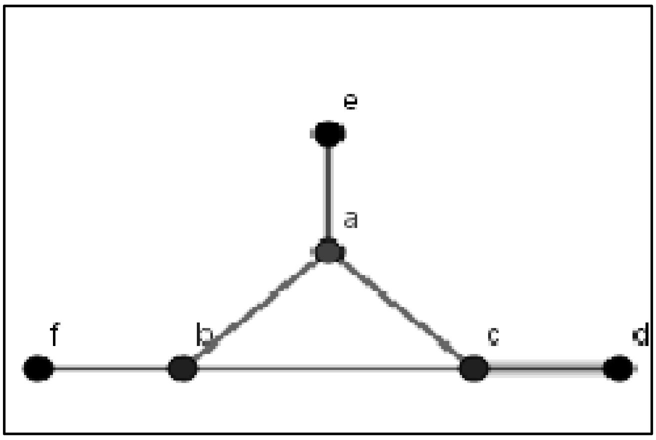

Figure 1. A graph which is not a divisor graph

2 Divisor graph of $\Gamma(Z\alpha(+)M)$

In this section, we give when $\Gamma(Z\alpha(+)M)$ is a divisor graph, where $Z\alpha = Z(+)Z$ is called the ring of dual numbers over the ring of integers. It is clear that $M \cong Z_n$ is a $Z\alpha$-module with the modulo operation defined by $(a+bx)m = am$.
We start with the following Lemma which was given in 3.

Lemma 1 *The zero divisor graph of* $Z\alpha(+)Z_n$, $Z^*(Z\alpha(+)Z_n) = V_1 \cup \quad V_2 \cup \quad V_3$, *where* $V_1 = \{(0,m)\colon m \in Z_n^*\}$. $V_2 = \{(a,t)\colon a \in Z^*\alpha, t \in Z_n^*$ and for some $m \in Z_n^*$ we have $am = 0\}$.

$$V_3 = \{(a,t)\colon a \in^* (Z\alpha), t \in Z_n\}$$

In our investigation, when $M \cong Z_{p_1p_2p_3}$, where p_1, p_2 and p_3 are three distinct primes.

Theorem 1 *If* M *is isomorphic to* $Z_{p_1p_2p_3}$, *then* $\Gamma(Z[\alpha](+)M)$ *is not a divisor graph.*

Proof. Assume that $M \cong Z_{p_1p_2p_3}$. Then we have the following induced subgraph of $\Gamma(Z\alpha(+)M)$ (Figure 1). Let $a = (0,p_2p_3)$, $d = (p_3,0)$, $b = (0,p_1p_3)$, $e = (p_1,0)$, $c = (0,p_1p_2)$ and $f = (p_2,0)$. Then $\Gamma(Z\alpha(+)M)$ is not a divisor graph.

Corollary 1 *If* M *is isomorphic to* $Z_{p_1^{\alpha_1}p_2^{\alpha_2}p_3^{\alpha_3}}$, $\alpha_1,\ \alpha_2,\ \alpha_3 \geq 2$, *then* $\Gamma(Z\alpha(+)M)$ *is not a divisor graph.*

Theorem 2 *If* M *is isomorphic to* $Z_{p_1^{\alpha_1}p_2^{\alpha_2}}$, $\alpha_1,\ \alpha_2 \geq 2$, *then* $\Gamma(Z\alpha(+)M)$ *is not a divisor graph.*

Proof. Assume that $M \cong Z_{p_1^{\alpha_1}p_2^{\alpha_2}}$. Then we have the following induced subgraph of $\Gamma(Z\alpha(+)M)$ (Figure 1). Let $a = (0,p_1^{\alpha_1-1}p_2^{\alpha_2-1})$, $d = (p_1^{\alpha_1},0)$, $b = (0,p_1^{\alpha_1})$, $e = (p_1p_2,0)$, $c = (0,p_2^{\alpha_2})$ and $f = (p_2^{\alpha_2},0)$. Then $\Gamma(Z\alpha(+)M)$ is not a divisor graph.

Theorem 3 *If* $M \cong Z_p$, *then* $\Gamma(Z\alpha(+)M)$ *is a divisor graph.*

Proof. If $M \cong Z_p$, then $\Gamma(Z\alpha(+)Z_p) = V_1 \cup \quad V_2 \cup \quad V_3$. Where $V_1 = \{(0,v)\colon v \in Z_p^*\}$, $V_2 = \{(bx,v)\colon b \in Z^*, v \in Z_p\}, V_3 = \{(pk+bx,v)\colon k,b \in Z^*, v \in Z_p\}$. Let $s_1, \ldots$, $q_1, \ldots$ and $l_1, \ldots$ be distinct odd primes. Then define the function $f\colon\ V(G) \to N$ by

$$f(x,y)=\begin{cases}\prod_{i=1}^{p} 2^i & , \quad (x,y)=(0,v_i)\\ 2^p\times\prod_{l=1}^{\infty} 5^l & , \quad (x,y)=(b_l x,v)\\ 2^p\times\prod_{i=1}^{\infty} S_i & , \quad (x,y)=(pk_i+bx,v)\end{cases}$$

Then f is a one to one function such that $(x,y)(\alpha,\beta)=(0,0)$ if and only if $f(x,y)$ divides $f(\alpha,\beta)$ or $f(\alpha,\beta)$ divides $f(x,y)$. Hence $\Gamma(Z\alpha(+)Z_p)$ is a divisor graph.

Theorem 4 *If $M\cong Z_{p^2}$, then $\Gamma(Z\alpha(+)M)$ is a divisor graph.*

Proof. If $M\cong Z_{p^2}$, then $\Gamma(Z\alpha(+)Z_{p^2})=V_1\cup\ V_2\cup\ V_3$. Where $V_1=\{(0,v):v\in Z^*_{p^2})\}, V_2=\{(bx,v):\ b\in Z^*,\ v\in Z_{p^2}\}, V_3=\{(pk+bx,v):k,\ b\in Z^*,\ v\in Z_{p^2}\}$. Let u be a unit in Z_{p^2} and v is an arbitrary element in Z_{p^2} and let $s_1^1,\dots,q_1,\dots$ be distinct odd primes. Then define the function $f:\ V(G)\to N$ by

$$f(x,y)=\begin{cases}\prod_{i=1}^{k} 2^i & , \quad (x,y)=(0,u_i)\\ 2^k\times\prod_{j=1}^{\infty} 3^j & , \quad (x,y)=(b_j x,v)\\ \prod_{m=1}^{\infty} S_m^j & , \quad (x,y)=(pk_m+b_j x,v)\\ 2^k\times\prod_{m=1}^{\infty} q_m & , \quad (x,y)=(p^2k_m+bx,v)\\ \prod_{r=1} 2^{k+r}\times\prod_{j=1}^{\infty} 3^j\times\prod_{m=1}^{\infty} q_m\times\prod_{m=1}^{\infty} S_m^j & , \quad (x,y)=(0,\quad pk_r)\end{cases}$$

Then f is a one to one function such that $(x,y)(\alpha,\beta)=(0,0)$ if and only if $f(x,y)$ divides $f(\alpha,\beta)$ or $f(\alpha,\beta)$ divides $f(x,y)$. Hence $\Gamma(Z[\alpha](+)Z_{p^2})$ is a divisor graph.

Corollary 2 *If $M\cong Z_{p^t} t\geq 2$, then $\Gamma(Z\alpha(+)M)$ is a divisor graph.*

Theorem 5 *If M isomorphic to $Z_{p_1p_2}$, then $\Gamma(Z\alpha(+)Z_{p_1p_2})$ is a divisor graph.*

Proof. If $M\cong Z_{p_1p_2}$, then $\Gamma(Z\alpha(+)Z_{p_1p_2})=V_1\cup\ V_2\cup\ V_3$. Where $V_1=\{(0,v):v\in Z^*_{p_1p_2},\ V_2=\{(bx,v):\ b\in Z^*,\ v\in Z_{p_1p_2}\}, V_3=\{(p_1k+bx,v),p_2k+bx,v):\ b,\ k\in Z^*,\ v\in Z_{p_1p_2}\}$. Let u is a unit in $Z_{p_1p_2}$, v arbitrary element in $Z_{p_1p_2}$, let $s_1^1,\dots$, $q_1^1,\dots$ and $l_1^1,\dots$ be distinct odd primes and v is arbitrary element in $Z_{p_1p_2}$. Then define the function $f:\ V(G)\to N$ by

$$f(x,y)=\begin{cases}\prod_{i=1}^{k} 2^i & , \quad (x,y)=(0,p_2k_i)\\ 2^k\times\prod_{j=1}^{m} 3^j & , \quad (x,y)=(0,u_j)\\ 2^k\times\ 3^m\times\prod_{l=1}^{\infty} 5^l & , \quad (x,y)=(b_l x,v)\\ 2^k\times\prod_{i=1}^{\infty} q_i^j & , \quad (x,y)=(p_1k_i+b_j x,v)\\ 2^i\prod_{i=1}^{\infty} s_i^j & , \quad (x,y)=(p_1p_2k_i+b_j x,v)\\ \prod_{i=1}^{\infty} l_i^j & , \quad (x,y)=(p_2k_i+b_j x,v)\\ 2^k\times 3^m\times\prod_{l=1}^{\infty} 5^l\times\prod_{i=1}^{\infty} q_i^j\times\prod_{i=1}^{\infty} s_i^j\times\prod_{i=1}^{\infty} l_i^j & , \quad (x,y)=(p_1k_i+b_j x,v)\end{cases}$$

Then f is a one to one function such that $(x,y)(\alpha,\beta)=(0,0)$ if and only if $f(x,y)$ divides $f(\alpha,\beta)$ or $f(\alpha,\beta)$ divides $f(x,y)$. Hence $\Gamma(Z\alpha(+)Z_{p_1p_2})$ is a divisor graph.

Theorem 6 *If M isomorphic to $Z_{p_1^2p_2}$, then $\Gamma(Z\alpha(+)Z_{p_1p_2})$ is a divisor graph.*
Proof. By the same procedure in the previous theorem.

3 Divisor graph of $\Gamma(Z_{p^{\alpha_1}}\alpha(+)Z_{p^{\alpha_2}})$

In this section, we give when $\Gamma(Z_{p^{\alpha_1}}\alpha(+)Z_{p^{\alpha_2}})$ is a divisor graph. It is clear that $Z_{p^{\alpha_1}}$ is a $Z_{p^{\alpha_2}}\alpha$-module with the modulo operation defined by $(a+bx)m=am$ if and only if $p^{\alpha_2}|p^{\alpha_1}$.
In our investigation, we start the following Lemma which was given in 3.

Lemma 2 *The zero divisor graph $Z^*(Z_m\alpha(+)Z_n)=V_1\cup\ V_2\cup\ V_3$, where $V_1=\{(0,m):m\in Z_n^*\}$.*
$V_2=\{(a,t):a\in Z_m^*\alpha, t\in Z_n^*$ and for some $m\in Z_n^*$ we have $am=0\}$.

$$V_3=\{(a,t):a\in\ Z^*(Z_m\alpha), t\in Z_n\}$$

Theorem 7 *If $n=m=p$, then $\Gamma(Z_p\alpha(+)Z_p)$ is a divisor graph.*

Proof. If $n=m=p$, then $\Gamma(Z_p\alpha(+)Z_p)=V_1\cup\ V_2\cup\ V_3$. Where $V_1=\{(0,v):v\in Z_p^*\}$, $V_2=\{(bx,v):b\in Z_m^*\alpha, v\in Z_p\}$. then $\Gamma(Z_p\alpha(+)Z_p)$ is complete graph which is a divisor graph.

Theorem 8 *If $m=p^2$ and $n=p$, then $\Gamma(Z_{p^2}\alpha(+)Z_p)$ is a divisor graph.*

Proof. If $m=p^2$ and $n=p$, then $\Gamma(Z_{p^2}\alpha(+)Z_p)=V_1\cup\ V_2\cup\ V_3$. Where $V_1=\{(0,v):v\in Z_p^*)\}, V_2=\{(bx,v):\ b\in Z_{p^2}^*,\ v\in Z_p\}, V_3=\{(pk+bx,v):k\in Z_{p^2}^*,\ b\in Z\ {}^{p^2}$ and $v\in Z_p\}$. Let $q_1^1,\dots$ be distinct odd primes. Then define the function $f\colon\ V(G)\to N$ by

$$f(x,y)=\begin{cases}\prod_{i=1}^{p}2^i & , \quad (x,y)=(0,v_i)\\ 2^p\times\prod_{j=1}^{m}3^j & , \quad (x,y)=(pk_jx,v)\\ 2^p\times 3^m\times\prod_{j=1}^{t}7^j & , \quad (x,y)=(b_jx,v)\\ 2^p\times 3^m\times\prod_{i=1}^{l}5^i & , \quad (x,y)=(pk_i+pkx,v)\\ 2^p\times 3^m\times\prod_{i=1}^{l}q_i^j & , \quad (x,y)=(pk_i+b_jx,v)\end{cases}$$

Then f is a one to one function such that $(x,y)(\alpha,\beta)=(0,0)$ if and only if $f(x,y)$ divides $f(\alpha,\beta)$ or $f(\alpha,\beta)$ divides $f(x,y)$. Hence $\Gamma(Z_{p^2}\alpha(+)Z_p)$ is a divisor graph.

Theorem 9 *If $m=p^2$ and $n=p^2$, then $\Gamma(Z_{p^2}\alpha(+)Z_{p^2})$ is a divisor graph.*

Proof. If $m=p^2$ and $n=p^2$, then $\Gamma(Z_{p^2}\alpha(+)Z_{p^2})=V_1\cup\ V_2\cup\ V_3$. Where $V_1=\{(0,v):v\in Z_{p^2}^*)\}, V_2=\{(bx,v):\ b\in Z_{p^2}^*,$ $v\in Z_{p^2}\}, V_3=\{(pk+bx,v):k\in Z_{p^2}^*,\ b\in Z\ {}^{p^2}$ and $v\in Z_{p^2}\}$. Let $u_i\neq pk_i$ and $b_j\neq pk_j$ are elements in Z_{p^2} and $q_1^1,\dots$ be distinct odd primes. Then define the function $f\colon\ V(G)\to N$ by

$$f(x,y)=\begin{cases}\prod_{i=1}^{k}2^{i} & , \quad (x,y)=(0,u_i)\\ 2^{k}\times\prod_{i=1}^{m}3^{i} & , \quad (x,y)=(0,pk_i)\\ 2^{k}\times 3^{m}\times\prod_{j=1}^{t}7^{j} & , \quad (x,y)=(p_jx,v)\\ 2^{k}\times 3^{m}\times 7^{t}\times\prod_{j=1}^{f}11^{j} & , \quad (x,y)=(b_jx,v)\\ 2^{p}\times 3^{m}\times 7^{t}\times\prod_{i=1}^{l}5^{i} & , \quad (x,y)=(pk_i+pkx,v)\\ 2^{p}\times 3^{m}\times\prod_{i=1}^{l}q_i^{j} & , \quad (x,y)=(pk_i+b_jx,v)\end{cases}$$

Then f is a one to one function such that $(x,y)(\alpha,\beta)=(0,0)$ if and only if $f(x,y)$ divides $f(\alpha,\beta)$ or $f(\alpha,\beta)$ divides $f(x,y)$. Hence $\Gamma(Z_{p^2}\alpha(+)Z_{p^2})$ is a divisor graph.

Corollary 3 *If $m=p^{\alpha_1}, n=p^{\alpha_2}$ where $\alpha_1\geq\alpha_2\geq 3$, then $\Gamma(Z_{p^{\alpha_1}}\alpha(+)Z_{p^{\alpha_2}})$ is a divisor graph.*

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