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RESEARCH ARTICLE

ON UNITARY BIMATRICES

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ABSTRACT

Unitary bimatrices are studied as a generalization of unitary matrices. Some of the properties of unitary matrices are extended to unitary bimatrices. Some important results of unitary matrices are generalized to unitary bimatrices.

INTRODUCTION

Matrices provide a very powerful tool for dealing with linear models. Bimatrices are still a powerful and an advanced tool which can handle over one linear model at a time. Bimatrices are useful when time bound comparisons are needed in the analysis of a model. Bimatrices are of several types. Unitary matrices are a firsthand tool in solving many problems in mathematical and theoretical physics and the diversity of the problems necessitates to keep improving it. For real matrices, unitary is the same as orthogonal .In fact there are some similarities between orthogonal matrices and unitary matrices. Here we consider all matrices belongs to $c_{n\times n}$. For any matrix A, A^* denotes the conjugate transpose of A. In this paper we have developed unitary bimatrices as a generalization of unitary matrices. Some of the properties of unitary matrices (Hari kishan 2008; Richard Bronson ?; Vatssa ?) are extended to unitary bimatrices. Some important results of unitary matrices (Rukmangadachari ?; Vasistha *et al.,* 2010) are generalized to unitary bimatrices.

Definition 1.1 (Vasantha Kandasamy *et al.*, 2005)

A bimatrix A_B is defined as the union of two rectangular array of numbers A_1 and A_2 arranged into rows and columns. It is written as $A_B = A_1 \cup A_2$ with $A_1 \neq A_2$ (except zero and unit bimatrices) where $A_1 = \begin{bmatrix} a_{11}^1 & a_{12}^1 & .... & a_{1n}^1 \\ a_{21}^1 & a_{22}^1 & .... & a_{2n}^1 \\ .... & .... & .... & .... \\ a_{m1}^1 & a_{m2}^1 & .... & a_{mn}^1 \end{bmatrix}$ and

$$A_2 = \begin{bmatrix} a_{11}^2 & a_{12}^2 & .... & a_{1n}^2 \\ a_{21}^2 & a_{22}^2 & .... & a_{2n}^2 \\ .... & .... & .... & .... \\ a_{m1}^2 & a_{m2}^2 & .... & a_{mn}^2 \end{bmatrix}$$

'$\cup$' is just the notational convenience (symbol) only.

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Definition 1.2 (Vasantha Kandasamy *et al.*, 2005)

Let $A_B = A_1 \cup A_2$ and $B_B = B_1 \cup B_2$ be any two $m \times n$ bimatrices. The sum C_B of the bimatrices A_B and B_B is defined as $C_B = A_B + B_B = (A_1 + B_1) \cup (A_2 + B_2)$, where $A_1 + B_1$ and $A_2 + B_2$ are the usual addition of matrices.

Definition 1.3 (Vasantha Kandasamy *et al.*, 2005)

If $A_B = A_1 \cup A_2$ and $B_B = B_1 \cup B_2$ are both $n \times n$ square bimatrices then, the bimatrix multiplication is defined as, $A_B \times B_B = (A_1 B_1) \cup (A_2 B_2)$.

Definition 1.4 (Vasantha Kandasamy *et al.*, 2005)

Let $A_B = A_1 \cup A_2$ and $B_B = B_1 \cup B_2$ be two bimatrices ,then A_B and B_B are said to be equal if and only if A_1 and B_1 are identical and A_2 and B_2 are identical. That is $A_1 = B_1$ and $A_2 = B_2$.

Definition 1.5 (Vasantha Kandasamy *et al.*, 2005)

If $A_B = A_1 \cup A_2$ is a $m \times m$ square bimatrix, then the identity bimatrix is defined as $I_B = I_1 \cup I_2$.

Remark 1.6 (Vasantha Kandasamy *et al.*, 2005)

If $A_B = A_1 \cup A_2$ be a bimatrix, then we call A_1 and A_2 as the component matrices of the bimatrix A_B.

II Unitary Bimatrices

In this section some of the properties of unitary matrices are extended to unitary bimatrices. Some important results of unitary matrices are generalized to unitary bimatrices.

Definition 2.1

Let $A_B = A_1 \cup A_2$ be an $n \times n$ complex bimatrix. (A bimatrix A_B is said to be complex if it takes entries from the complex field). A_B is called an unitary bimatrix if $A_B A_B^* = A_B^* A_B = I_B$ (or) $\overline{A_B}^T = A_B^{-1}$. That is, $A_1 A_1^* \cup A_2 A_2^* = A_1^* A_1 \cup A_2^* A_2 = I_1 \cup I_2$.

Example 2.2

Let $A_B = A_1 \cup A_2$

$$A_B = \frac{1}{\sqrt{2}}\begin{bmatrix} i & i \\ i & -i \end{bmatrix} \cup \frac{1}{2}\begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix}; \quad A_B^* = \frac{1}{\sqrt{2}}\begin{bmatrix} -i & -i \\ -i & i \end{bmatrix} \cup \frac{1}{2}\begin{bmatrix} 1-i & 1-i \\ -1-i & 1+i \end{bmatrix}$$

$$A_B A_B^* = \left(\frac{1}{\sqrt{2}}\begin{bmatrix} i & i \\ i & -i \end{bmatrix} \times \frac{1}{\sqrt{2}}\begin{bmatrix} -i & -i \\ -i & i \end{bmatrix} \right) \cup \left(\frac{1}{2}\begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix} \times \frac{1}{2}\begin{bmatrix} 1-i & 1-i \\ -1-i & 1+i \end{bmatrix} \right)$$

$$A_B A_B^* = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cup \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_1 \cup I_2$$

$$A_B A_B^* = I_B \tag{1}$$

Similarly, we can find that $A_B^* A_B = I_B$ (2)

From (1) and (2), we get $A_B A_B^* = A_B^* A_B = I_B$.

Hence, A_B is a unitary bimatrix.

Theorem 2.3

Product of two unitary bimatrices of the same order is a unitary bimatrix.

Proof

Let $A_B = A_1 \cup A_2$ and $B_B = B_1 \cup B_2$ be unitary bimatrices so that $A_1 A_1^* \cup A_2 A_2^* = A_1^* A_1 \cup A_2^* A_2 = I_1 \cup I_2$; $B_1 B_1^* \cup B_2 B_2^* = B_1^* B_1 \cup B_2^* B_2 = I_1 \cup I_2$; (or) $\overline{A}_B^{\,T} = A_B^{-1}; \overline{B}_B^{\,T} = B_B^{-1}$.

$$
\begin{aligned}
\text{Now } (A_B B_B)(A_B B_B)^* &= \left[(A_1 \cup A_2)(B_1 \cup B_2)\right]\left[(A_1 \cup A_2)(B_1 \cup B_2)\right]^* \\
&= \left[A_1 B_1 \cup A_2 B_2\right]\left[A_1 B_1 \cup A_2 B_2\right]^* \\
&= \left[A_1 B_1 \cup A_2 B_2\right]\left[(A_1 B_1)^* \cup (A_2 B_2)^*\right] \\
&= \left[A_1 B_1 \cup A_2 B_2\right]\left[B_1^* A_1^* \cup B_2^* A_2^*\right] \\
&= \left[A_1 \left(B_1 B_1^*\right) A_1^*\right] \cup \left[A_2 \left(B_2 B_2^*\right) A_2^*\right] \\
&= \left(A_1 I A_1^*\right) \cup \left(A_2 I A_2^*\right) \\
&= \left(A_1 A_1^*\right) \cup \left(A_2 A_2^*\right) \\
&= I_1 \cup I_2
\end{aligned}
$$

$(A_B B_B)(A_B B_B)^* = I_B$ (3)

Similarly, we can prove that $(A_B B_B)^* (A_B B_B) = I_B$ (4)

From (3) and (4) we get, $(A_B B_B)\ (A_B B_B)^* = (A_B B_B)^* (A_B B_B) = I_B$

Hence, the product of two unitary bimatrices is a unitary bimatrrix.

Example 2.4

Let $A_B = \frac{1}{\sqrt{7}}\begin{bmatrix} 1+2i & 1+i \\ 1-i & -1+2i \end{bmatrix} \cup \frac{1}{\sqrt{2}}\begin{bmatrix} i & 1 \\ -i & 1 \end{bmatrix}$ and $B_B = \frac{1}{\sqrt{2}}\begin{bmatrix} i & i \\ i & -i \end{bmatrix} \cup \frac{1}{2}\begin{bmatrix} i & \sqrt{3} \\ \sqrt{3} & i \end{bmatrix}$

$$
(A_B B_B) = \left(\frac{1}{\sqrt{7}}\begin{bmatrix} 1+2i & 1+i \\ 1-i & -1+2i \end{bmatrix} \times \frac{1}{\sqrt{2}}\begin{bmatrix} i & i \\ i & -i \end{bmatrix}\right) \cup \left(\frac{1}{\sqrt{2}}\begin{bmatrix} i & 1 \\ -i & 1 \end{bmatrix} \times \frac{1}{2}\begin{bmatrix} i & \sqrt{3} \\ \sqrt{3} & i \end{bmatrix}\right)
$$

$$
(A_B B_B) = \frac{1}{\sqrt{14}}\begin{bmatrix} -3+2i & -1 \\ -1 & 3+2i \end{bmatrix} \cup \frac{1}{2\sqrt{2}}\begin{bmatrix} -1+\sqrt{3} & i\sqrt{3}+i \\ 1+\sqrt{3} & -i\sqrt{3}+i \end{bmatrix}
$$

$$\left(A_B B_B\right)^* = \frac{1}{\sqrt{14}}\begin{bmatrix} -3-2i & -1 \\ -1 & 3-2i \end{bmatrix} \cup \frac{1}{2\sqrt{2}}\begin{bmatrix} -1+\sqrt{3} & \sqrt{3}+1 \\ -i\sqrt{3}-i & i\sqrt{3}-i \end{bmatrix}$$

$$\left(A_B B_B\right)\left(A_B B_B\right)^*$$

$$= \frac{1}{\sqrt{14}}\left(\begin{bmatrix} -3+2i & -1 \\ -1 & 3+2i \end{bmatrix} \times \begin{bmatrix} -3-2i & -1 \\ -1 & 3-2i \end{bmatrix}\right) \cup \frac{1}{2\sqrt{2}}\left(\begin{bmatrix} -1+\sqrt{3} & i\sqrt{3}+i \\ 1+\sqrt{3} & -i\sqrt{3}+i \end{bmatrix} \times \begin{bmatrix} -1+\sqrt{3} & \sqrt{3}+1 \\ -i\sqrt{3}-i & i\sqrt{3}-i \end{bmatrix}\right)$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cup \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_1 \cup I_2$$

$$\left(A_B B_B\right)\left(A_B B_B\right)^* = I_B \qquad (5)$$

Similarly, we can find that $(A_B B_B)^* \left(A_B B_B\right) = I_B$ (6)

From (5) and (6) we get, $(A_B B_B)\ (A_B B_B)^* = (A_B B_B)^* (A_B B_B) = I_B$

Hence, $A_B B_B$ is a unitary bimatrix.

Theorem 2.5

Inverse of a unitary bimatrix is a unitary bimatrix.

Proof

For a unitary bimatrix A_B, $A_B A_B^* = A_B^* A_B = I_B$ or $\overline{A}_B^{\,T} = A_B^{-1}$.

$$\begin{aligned} A_B^{-1}\left(A_B^{-1}\right)^* &= \left(A_1 \cup A_2\right)^{-1}\left[\left(A_1 \cup A_2\right)^{-1}\right]^* \\ &= A_1^{-1}\left(A_1^{-1}\right)^* \cup A_2^{-1}\left(A_2^{-1}\right)^* \\ &= \left(A_1^{-1} \cup A_2^{-1}\right)\left[A_1^{-1} \cup A_2^{-1}\right]^* \\ &= \left(A_1^{-1} \cup A_2^{-1}\right)\left[\left(A_1^{-1}\right)^* \cup \left(A_2^{-1}\right)^*\right] \\ &= A_1^{-1}\left(A_1^{-1}\right)^* \cup A_2^{-1}\left(A_2^{-1}\right)^* \\ &= I_1 \cup I_2 \end{aligned}$$

$$(A_B^{-1})(A_B^{-1})^* = I_B \qquad (7)$$

Similarly, we can prove that $(A_B^{-1})^* \left(A_B^{-1}\right) = I_B$ (8)

From (7) and (8), we get $A_B^{-1}\left(A_B^{-1}\right)^* = (A_B^{-1})^* \left(A_B^{-1}\right) = I_B$.

Hence, A_B^{-1} is a unitary bimatrrix.

Example 2.6

Let $A_B = A_1 \cup A_2 = \frac{1}{\sqrt{5}}\begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix} \cup \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ -i & -1 \end{bmatrix}$

$$|A_B| = \frac{1}{\sqrt{5}}\begin{vmatrix} 0 & 1+2i \\ -1+2i & 0 \end{vmatrix} \cup \frac{1}{\sqrt{2}}\begin{vmatrix} 1 & i \\ -i & -1 \end{vmatrix}$$

$$= 1 \cup 1$$

$$A_B^{-1} = \frac{1}{\sqrt{5}}\begin{bmatrix} 0 & -1-2i \\ 1-2i & 0 \end{bmatrix} \cup \frac{1}{\sqrt{2}}\begin{bmatrix} -1 & -i \\ i & 1 \end{bmatrix}$$

$$\left(A_B^{-1}\right)\left(A_B^{-1}\right)^* = \frac{1}{\sqrt{5}}\left(\begin{bmatrix} 0 & -1-2i \\ 1-2i & 0 \end{bmatrix} \times \begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix}\right) \cup \frac{1}{\sqrt{2}}\left(\begin{bmatrix} -1 & -i \\ i & 1 \end{bmatrix} \times \begin{bmatrix} -1 & -i \\ i & 1 \end{bmatrix}\right)$$

$$\left(A_B^{-1}\right)\left(A_B^{-1}\right)^* = \frac{1}{\sqrt{5}}\left(\begin{bmatrix} 0 & -1-2i \\ 1-2i & 0 \end{bmatrix} \times \begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix}\right) \cup \frac{1}{\sqrt{2}}\left(\begin{bmatrix} -1 & -i \\ i & 1 \end{bmatrix} \times \begin{bmatrix} -1 & -i \\ i & 1 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cup \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} == I_1 \cup I_2$$

$$(A_B^{-1})(A_B^{-1})^* = I_B \tag{9}$$

Similarly, we can find that $(A_B^{-1})^*\left(A_B^{-1}\right) = I_B$ (10)

From (9) and (10) we get, $A_B^{-1}\left(A_B^{-1}\right)^* = (A_B^{-1})^*\left(A_B^{-1}\right) = I_B$.

Hence, A_B^{-1} is a unitary bimatrix.

Theorem 2.7

Transpose of a unitary bimatrix is a unitary bimatrix.

Proof

For a unitary bimatrices $A_B, A_B A_B^* = A_B^* A_B = I_B$ (or) $\overline{A}_B{}^T = \overline{A}_B{}^{-1}$

$$\begin{aligned} \left(A_B^T\right)\left(A_B^T\right)^* &= \left(A_1 \cup A_2\right)^T \times \left[\left(A_1 \cup A_2\right)^T\right]^* \\ &= \left(A_1^T \cup A_2^T\right) \times \left[\left(A_1^T\right)^* \cup \left(A_2^T\right)^*\right] \\ &= \left[\left(A_1^T\right)\left(A_1^T\right)^*\right] \cup \left[\left(A_2^T\right)\left(A_2^T\right)^*\right] = I_1 \cup I_2 \end{aligned}$$

$$\left(A_B^T\right)\left(A_B^T\right)^* = I_B \tag{11}$$

Similarly, we can prove that $\left(A_B^T\right)^*\left(A_B^T\right) = I_B$ (12)

From (11) and (12), we get $\left(A_B^T\right)\left(A_B^T\right)^* = \left(A_B^T\right)^*\left(A_B^T\right) = I_B$.

Hence A_B^T is a unitary bimatrix .

Example 2.8

Let $A_B = A_1 \cup A_2 = \frac{1}{2}\begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix} \cup \frac{1}{2}\begin{bmatrix} i & \sqrt{3} \\ \sqrt{3} & i \end{bmatrix}$

$$A_B^T = \frac{1}{2}\begin{bmatrix} 1+i & 1+i \\ -1+i & 1-i \end{bmatrix} \cup \frac{1}{2}\begin{bmatrix} i & \sqrt{3} \\ \sqrt{3} & i \end{bmatrix}$$

$$\left(A_B^T\right)^* = \frac{1}{2}\begin{bmatrix} 1-i & -1-i \\ 1-i & 1+i \end{bmatrix} \cup \frac{1}{2}\begin{bmatrix} -i & \sqrt{3} \\ \sqrt{3} & -i \end{bmatrix}$$

$$\left(A_B^T\right)\left(A_B^T\right)^* = \left(\frac{1}{2}\begin{bmatrix} 1+i & 1+i \\ -1+i & 1-i \end{bmatrix} \times \frac{1}{2}\begin{bmatrix} 1-i & -1-i \\ 1-i & 1+i \end{bmatrix}\right) \cup \left(\frac{1}{2}\left[\begin{bmatrix} i & \sqrt{3} \\ \sqrt{3} & i \end{bmatrix} \times \frac{1}{2}\begin{bmatrix} i & \sqrt{3} \\ \sqrt{3} & i \end{bmatrix}\right]\right)$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cup \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_1 \cup I_2$$

$$\left(A_B^T\right)\left(A_B^T\right)^* = I_B \qquad (13)$$

Similarly, we can find that $\left(A_B^T\right)^*\left(A_B^T\right) = I_B$ (14)

From (13) and (14), we get $\left(A_B^T\right)\left(A_B^T\right)^* = \left(A_B^T\right)^*\left(A_B^T\right) = I_B$.

Hence A_B^T is a unitary bimatrix.

Theorem 2.9

The determinant of a unitary bimatrix has absolute value 1.

Proof

If A_B is a unitary bimatrix then we have $A_B A_B^* = A_B^* A_B = I_B$ (or) $\overline{A_B}^T = \overline{A_B}^{-1}$.

$$det\left(A_B A_B^*\right) = det\left(A_B A_B^{-1}\right) = \det\left(I_B\right)$$

$$det\left(A_B \overline{A}_B^T\right) = 1 \cup 1$$

$$\det A_B \det \overline{A}_B^T = 1 \cup 1$$

$$\det A_B \det \overline{A}_B = 1 \cup 1$$

$$\det A_B \, \overline{\det A_B} = 1 \cup 1$$

$\left|\det A_B\right|^2 = 1 \Rightarrow \left|\det A_B\right| = 1$ (Where $\det A_B$ may now be complex)

Hence the determinant of a unitary bimatrix has absolute value 1.

Example 2.10

Let $A_B = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ -i & -1 \end{bmatrix} \cup \frac{1}{2}\begin{bmatrix} 1 & i\sqrt{3} \\ i\sqrt{3} & 1 \end{bmatrix}$

$$\left|A_B\right| = \frac{1}{\sqrt{2}}\begin{vmatrix} 1 & i \\ -i & -1 \end{vmatrix} \cup \frac{1}{2}\begin{vmatrix} 1 & i\sqrt{3} \\ i\sqrt{3} & 1 \end{vmatrix} = \left|-1\right| \cup \left|1\right| = 1 \cup 1$$

Hence, $\left|A_B\right| = 1$.

Theorem 2.11

Conjugate of a unitary bimatrix is a unitary bimatrix.

Proof

Let $A_B = A_1 \cup A_2$ be unitary bimatrix. That is , $A_B A_B^* = A_B^* A_B = I_B$ (or) $\overline{A}_B{}^T = A_B^{-1}$.

$$A_B = A_1 \cup A_2$$

$$A_B A_B^* = (A_1 \cup A_2)(A_1 \cup A_2)^*$$

$$A_B A_B^* = (A_1 \cup A_2)(A_1^* \cup A_2^*)$$

$$A_B A_B^* = (A_1 A_1^* \cup A_2 A_1^*)$$

Taking conjugate on both sides,

$$\overline{A_B A_B^*} = \overline{A_1 A_1^* \cup A_2 A_2^*}$$

$$\overline{A}_B \overline{A}_B^* = \overline{I}_B \cup \overline{I}_B$$

$$= I_1 \cup I_2$$

$$\overline{A}_B \left(\overline{A}_B\right)^* = I_B \tag{15}$$

Similarly, we can find that $\left(\overline{A}_B\right)^* \left(\overline{A}_B\right) = I_B$ (16)

From (15) and (16), we get $\overline{A}_B \left(\overline{A}_B\right)^* = \left(\overline{A}_B\right)^* \overline{A}_B = I_B$

Hence, $\overline{A}_B$ is a unitary bimatrix.

Example 2.12

Let $A_B = \frac{1}{\sqrt{2}}\begin{bmatrix} i & 1 \\ -i & 1 \end{bmatrix} \cup \frac{1}{\sqrt{5}}\begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix}$

$$\overline{A}_B = \frac{1}{\sqrt{2}}\begin{bmatrix} -i & 1 \\ i & 1 \end{bmatrix} \cup \frac{1}{\sqrt{5}}\begin{bmatrix} 0 & 1-2i \\ -1-2i & 0 \end{bmatrix}$$

$$\overline{A}_B \overline{A}_B^* = \left(\frac{1}{\sqrt{2}}\begin{bmatrix} -i & 1 \\ i & 1 \end{bmatrix} \times \frac{1}{\sqrt{2}}\begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix} \right) \cup \left(\frac{1}{\sqrt{5}}\begin{bmatrix} 0 & 1-2i \\ -1-2i & 0 \end{bmatrix} \times \frac{1}{\sqrt{5}}\begin{bmatrix} 0 & -1+2i \\ 1+2i & 0 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cup \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_1 \cup I_2$$

$$\overline{A}_B \left(\overline{A}_B \right)^* = I_B \tag{17}$$

Similarly, we can find that $\left(A_B^T \right)^* \left(\overline{A}_B \right) = I_B$. $\quad (18)$

From (17) and (18) ,we get $\overline{A}\left(\overline{A}_B \right)^* = \left(\overline{A}_B \right)^* \overline{A} = I_B$

Hence $\overline{A}_B$ is a unitary bimatrix .

Theorem 2.13

Conjugate transpose of a unitary bimatrix is a unitary bimatrix.

Proof

Let $A_B = A_1 \cup A_2$ be unitary bimatrix. That is, $A_B A_B^* = A_B^* A_B = I_B$ (or) $\overline{A}_B{}^T = A_B^{-1}$.

Consider $A_B A_B^* = (A_1 \cup A_2)(A_1 \cup A_2)^*$

$$A_B A_B^* = (A_1 \cup A_2)(A_1^* \cup A_2^*)$$

$$A_B A_B^* = (A_1 A_1^* \cup A_2 A_2^*)$$

Taking conjugate transpose on both sides,

$$\left(A_B A_B^* \right)^* = \left(A_1 A_1^* \cup A_2 A_2^* \right)^*$$

$$(A_B^*)^* A_B^* = (A_1 A_1^*)^* \cup \left(A_2 A_2^* \right)^*$$

$$= I_1 \cup I_2$$

$$A_B A_B^* = I_B \tag{19}$$

Similarly, we can prove that $A_B^* A_B = I_B$ $\quad (20)$

From (19) and (20), we get $A_B A_B^* = A_B^* A_B = I_B$.

Hence, A_B^* is a unitary bimatrix.

Example 2.14

Let $A_B = A_1 \cup A_2 = \frac{1}{2}\begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix} \cup \frac{1}{2}\begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix}$

$$A_B^* = \frac{1}{2}\begin{bmatrix} 1-i & 1+i \\ 1+i & 1-i \end{bmatrix} \cup \frac{1}{2}\begin{bmatrix} 1-i & 1-i \\ -1-i & 1+i \end{bmatrix}$$

$$A_B^*\left(A_B^*\right)^* = A_B^* A_B = \left(\frac{1}{2}\begin{bmatrix} 1-i & 1+i \\ 1+i & 1-i \end{bmatrix}\right) \times \left(\frac{1}{2}\begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix}\right) \cup \left(\frac{1}{2}\begin{bmatrix} 1-i & 1-i \\ -1-i & 1+i \end{bmatrix}\right) \times \left(\frac{1}{2}\begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix}\right)$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cup \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_1 \cup I_2$$

$$A_B^*\left(A_B^*\right)^* = I_B \qquad (21)$$

Similarly, we can find that $(A_B^*)^* A_B^* = I_B$ (22)

From (21) and (22), we get $A_B^*\left(A_B^*\right)^* = \left(A_B^*\right)^* A_B^* = I_B$

Hence, A_B^* is a unitary bimatrix.

Theorem 2.15

Any integral power of a unitary bimatrix is also a unitary bimatrix.

Proof

Let $A_B = A_1 \cup A_2$ be unitary bimatrix. That is, $A_B A_B^* = A_B^* A_B = I_B$ (or) $\overline{A_B}^T = A_B^{-1}$.

Consider $A_B A_B^* = (A_1 \cup A_2)(A_1 \cup A_2)^*$

$$= (A_1 \cup A_2)(A_1^* \cup A_2^*)$$
$$= (A_1 A_1^* \cup A_2 A_2^*)$$
$$= I_1 \cup I_2$$

$$A_B A_B^* = I_B \qquad (23)$$

Similarly, we can prove that $A_B^* A_B = I_B$.

Again, $\left(A_B A_B^*\right)^2 = \left(A_B A_B^*\right)\left(A_B A_B^*\right)$

$$= \left[(A_1 \cup A_2)(A_1 \cup A_2)^*\right]\left[(A_1 \cup A_2)(A_1 \cup A_2)^*\right]$$

$$=\left[(A_1\cup A_2)(A_1^*\cup A_2^*)\right]\left[(A_1\cup A_2)(A_1^*\cup A_2^*)\right]$$
$$=\left[(A_1A_1^*\cup A_2A_2^*)\right]\left[(A_1A_1^*\cup A_2A_2^*)\right]$$
$$=\left[I_1\cup I_2\right]\left[I_1\cup I_2\right]$$
$$=I_1\cup I_2$$

$$\left(A_BA_B^*\right)^2=I_B$$

Similarly, we can prove that $\left(A_BA_B^*\right)^2=I_B$.

Hence, A_B^2 is a unitary bimatrix.

Assume that A_B^k is a unitary bimatrix. That is, $\left(A_BA_B^*\right)^k=\left(A_B^*A_B\right)^k=I_B$ (24)

To prove A_B^{k+1} is a unitary bimatrix.

$$\left(A_BA_B^*\right)^{k+1}=\left(A_BA_B^*\right)\left(A_BA_B^*\right)^k$$
$$=I_B.I_B \quad \text{(Since by (23) and (24))}$$
$$=I_B^2$$
$$\left(A_BA_B^*\right)^{k+1}=I_B$$

Similarly, we can prove that $\left(A_B^*A_B\right)^{k+1}=I_B$.

Hence, any integral power of a unitary bimatrix is also a unitary bimatrix.

Example 2.16

Let $A_B=A_1\cup A_2=\frac{1}{2}\begin{bmatrix} i & \sqrt{3} \\ \sqrt{3} & i\end{bmatrix}\cup\frac{1}{2}\begin{bmatrix} i & 1 \\ -i & 1\end{bmatrix}$

$$\left(A_B\right)^2=A_BA_B=\left(\frac{1}{2}\begin{bmatrix} i & \sqrt{3} \\ \sqrt{3} & i\end{bmatrix}\times\frac{1}{2}\begin{bmatrix} i & \sqrt{3} \\ \sqrt{3} & i\end{bmatrix}\right)\cup\left(\frac{1}{2}\begin{bmatrix} i & 1 \\ -i & 1\end{bmatrix}\times\frac{1}{2}\begin{bmatrix} i & 1 \\ -i & 1\end{bmatrix}\right)$$

$$\left(A_B\right)^2=\frac{1}{2}\begin{bmatrix} 1 & i\sqrt{3} \\ i\sqrt{3} & 1\end{bmatrix}\cup\frac{1}{2}\begin{bmatrix} -1-i & 1+i \\ 1-i & 1-i\end{bmatrix}$$

$$\left(A_B^2\right)^*=\frac{1}{2}\begin{bmatrix} 1 & -i\sqrt{3} \\ -i\sqrt{3} & 1\end{bmatrix}\cup\frac{1}{2}\begin{bmatrix} -1+i & 1+i \\ 1-i & 1+i\end{bmatrix}$$

$$A_B^2\left(A_B^2\right)^*=\left(\frac{1}{2}\begin{bmatrix} 1 & i\sqrt{3} \\ i\sqrt{3} & 1\end{bmatrix}\times\frac{1}{2}\begin{bmatrix} 1 & -i\sqrt{3} \\ -i\sqrt{3} & 1\end{bmatrix}\right)\cup\left(\frac{1}{2}\begin{bmatrix} -1-i & 1+i \\ 1-i & 1-i\end{bmatrix}\times\frac{1}{2}\begin{bmatrix} -1+i & 1+i \\ 1-i & 1+i\end{bmatrix}\right)$$

$$=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\cup\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=I_1\cup I_2$$

$$A_B^2\left(A_B^2\right)^*=I_B \qquad (25)$$

Similarly, we can find that $\left(A_B^2\right)^* A_B^2=I_B$ $\qquad (26)$

From (25) and (26), we get $A_B^2\left(A_B^2\right)^*=\left(A_B^2\right)^* A_B^2=I_B$.

Hence, A_B^2 is a unitary bimatrix.

Remark 2.17

Powers of unitary bimatrices occurring in applications may sometimes be familiar real matrices.

Example 2.18

Let

$$A_B=A_1\cup A_2=\frac{1}{2}\begin{bmatrix}1 & i\sqrt{3}\\ i\sqrt{3} & 1\end{bmatrix}\cup\frac{1}{2}\begin{bmatrix}1+i & -1+i\\ 1+i & 1-i\end{bmatrix}$$

$$A_B^2=\left(\frac{1}{2}\begin{bmatrix}-1 & i\sqrt{3}\\ i\sqrt{3} & -1\end{bmatrix}\times\frac{1}{2}\begin{bmatrix}1 & i\sqrt{3}\\ i\sqrt{3} & 1\end{bmatrix}\right)\cup\left(\left[\frac{1}{2}\begin{bmatrix}1+i & -1+i\\ 1+i & 1-i\end{bmatrix}\right]\times\left[\frac{1}{2}\begin{bmatrix}1+i & -1+i\\ 1+i & 1-i\end{bmatrix}\right]\right)$$

$$A_B^2=\frac{1}{2}\begin{bmatrix}1 & i\sqrt{3}\\ i\sqrt{3} & 1\end{bmatrix}\cup\frac{1}{4}\begin{bmatrix}-2+2i & -2+2i\\ 2+2i & -2-2i\end{bmatrix}$$

$$A_B^3=A_B^2.A^2=\left(\frac{1}{2}\begin{bmatrix}-1 & i\sqrt{3}\\ i\sqrt{3} & -1\end{bmatrix}\times\frac{1}{2}\begin{bmatrix}1 & i\sqrt{3}\\ i\sqrt{3} & 1\end{bmatrix}\right)\cup\left(\frac{1}{4}\begin{bmatrix}-2+2i & -2+2i\\ 2+2i & -2-2i\end{bmatrix}\times\frac{1}{2}\begin{bmatrix}1+i & -1+i\\ 1+i & 1-i\end{bmatrix}\right)$$

$$A_B^3=\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}\cup\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}=-I_1\cup\left(-I_2\right)=-I_B$$

Hence, $A_B^3=-I_B$.

Theorem 2.19

Unitary bimatrices are normal.

Proof

Let $A_B=A_1\cup A_2$ be unitary bimatrix. That is, $A_B A_B{}^*=A_B{}^* A_B=I_B$ (or) $\overline{A}_B^T=A_B^{-1}$.

Consider $A_B A_B^*=(A_1\cup A_2)(A_1\cup A_2)^*$

$$A_B A_B^*=(A_1\cup A_2)(A_1^*\cup A_2^*)$$

$$A_B A_B^* = (A_1 A_1^* \cup A_2 A_2^*) \quad = I_B^1 \cup I_B^2$$
$$A_B^* A_B = \left(A_1 \cup A_2\right)^* \left(A_1 \cup A_2\right)$$
$$A_B^* A_B = \left(A_1^* \cup A_2^*\right)\left(A_1 \cup A_2\right)$$
$$A_B^* A_B = A_1^* A_1 \cup A_2^* A_2$$
$$A_B^* A_B = I_1 \cup I_2$$

$$A_B A_B^* = I_B \tag{27}$$

Similarly we can find that $A_B^* A_B = I_B$ (28)

From (27) and (28), we get $A_B A_B^* = A_B^* A_B = I_B$

Hence unitary bimatrices are normal.

Example 2.20

Let $A_B = A_1 \cup A_2 = = \frac{1}{\sqrt{5}}\begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix} \cup \frac{1}{\sqrt{7}}\begin{bmatrix} 1+2i & 1+i \\ 1-i & -1+2i \end{bmatrix}$

$$A_B^* = \frac{1}{\sqrt{5}}\begin{bmatrix} 0 & -1-2i \\ 1-2i & 0 \end{bmatrix} \cup \frac{1}{\sqrt{7}}\begin{bmatrix} 1-2i & 1+i \\ 1-i & -1-2i \end{bmatrix}$$

$$A_B A_B^* = \frac{1}{\sqrt{5}}\left(\begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix} \times \begin{bmatrix} 0 & -1-2i \\ 1-2i & 0 \end{bmatrix}\right) \cup \frac{1}{\sqrt{7}}\left(\begin{bmatrix} 1+2i & 1+i \\ 1-i & -1+2i \end{bmatrix} \times \begin{bmatrix} 1-2i & 1+i \\ 1-i & -1-2i \end{bmatrix}\right)$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cup \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_1 \cup I_2$$

$$A_B A_B^* = I_B \tag{29}$$

$$A_B^* A_B = \frac{1}{\sqrt{5}}\left(\begin{bmatrix} 0 & -1-2i \\ 1-2i & 0 \end{bmatrix} \times \begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix}\right) \cup \frac{1}{\sqrt{7}}\left(\begin{bmatrix} 1-2i & 1+i \\ 1-i & -1-2i \end{bmatrix} \times \begin{bmatrix} 1+2i & 1+i \\ 1-i & -1-2i \end{bmatrix}\right)$$

$$A_B^* A_B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cup \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_1 \cup I_2$$

$$A_B^* A_B = I_B \tag{30}$$

From (29) and (30), we get $A_B A_B^* = A_B^* A_B = I_B$

Hence, A_B is normal.

Result 2.21

Sum of two unitary bimatrices need not be a unitary bimatrix.

Example 2.22

Let

$$A_B = \frac{1}{2}\begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix} \cup \frac{1}{2}\begin{bmatrix} i & \sqrt{3} \\ \sqrt{3} & i \end{bmatrix}; \quad B_B = \frac{1}{2}\begin{bmatrix} 1 & i\sqrt{3} \\ i\sqrt{3} & 1 \end{bmatrix} \cup \frac{1}{2}\begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix}$$

$$A_B + B_B = \frac{1}{2}\begin{bmatrix} 2+i & 1-i+i\sqrt{3} \\ 1-i+i\sqrt{3} & 2+i \end{bmatrix} \cup \frac{1}{2}\begin{bmatrix} 1+2i & -1+\sqrt{3}+i \\ 1+\sqrt{3}+i & 1 \end{bmatrix}$$

$$\left(A_B + B_B\right)^* = \frac{1}{2}\begin{bmatrix} 2-i & 1+i-i\sqrt{3} \\ 1+i-i\sqrt{3} & 2-i \end{bmatrix} \cup \frac{1}{2}\begin{bmatrix} 1-2i & 1+\sqrt{3}-i \\ -1+\sqrt{3}-i & 1 \end{bmatrix}$$

$$\left(A_B + B_B\right)\left(A_B + B_B\right)^* = \frac{1}{4}\begin{bmatrix} 10-2\sqrt{3} & 2+2\sqrt{3} \\ 2+2\sqrt{3} & 10-2\sqrt{3} \end{bmatrix} \cup \frac{1}{4}\begin{bmatrix} 10-2\sqrt{3} & 2+2\sqrt{3}+2+2i\sqrt{3} \\ 2+2\sqrt{3}-2-2i\sqrt{3} & 6+2\sqrt{3} \end{bmatrix}$$

$$\left(A_B + B_B\right)\left(A_B + B_B\right)^* \neq I \qquad (31)$$

Similarly, we can find that $\left(A_B + B_B\right)^*\left(A_B + B_B\right) \neq I$ (32)

From (31) and (32), we get $\left(A_B + B_B\right)\left(A_B + B_B\right)^* = \left(A_B + B_B\right)^*\left(A_B + B_B\right) \neq I$.

Hence, $A_B + B_B$ is not a unitary bimatrix.

Conclusion

Some of the properties of unitary matrices are proved for unitary bimatrices. In a similar way all the properties of unitary matrices can be verified for unitary bimatrices.

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