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RESEARCH ARTICLE

A NOTE ON THE CONTINUABILITY OF SOME LIENARD'S TYPE SYSTEM

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ABSTRACT

In this note we obtain sufficient conditions under which we can guarantee the continuability of solutions of a system which contain the classical Liénard equation.

INTRODUCTION

The classical method of Lyapunov for studying stability and asymptotic stability is based in a suitable function satisfying some properties (called Lyapunov Functions). This method, usually named Direct or Second Method, was originated in the fundamental memoir of the russian mathematician Alexander Mijailovich Lyapunov, published in russian in 1892, translated into french in 1907 (reprinted in the forty) and in english years later, see (Liapounov, 1949). Since that time this area has been extensively (perhaps even exhaustively) investigated. Statements and proofs of mathematical results underlying the method and numerous examples and references can be found in the books of Antoisewicz (1958), Barbashin (1970), Cesari (1959), Demidovich (1967), Hahn (1963), Yoshizawa (1966) and bibliography listed therein.

The problem of the continuability of solutions is of paramount importance in the study of qualitative properties. For example, in Nápoles Valdés (1995) some properties of the system

$$\begin{cases} \dot{x} = \alpha(y) - \beta(y)F(x) \\ \dot{y} = -g(x) \end{cases} \quad \ldots\ldots(1)$$

was studied under suitable assumptions. A particular case of above system is the well known Lienard equation

$$\ddot{x} + f(x)\dot{x} + g(x) = 0 \quad \ldots..(2)$$

where f(x), g(x) are continuous functions (f, g : R → R) and g(0) = 0, is the subject of detailed studies of many authors due to many applications in various domains in science and tecnology, see for example (Hara *et al.,* 1985; Hasan *et al.,* 2007; Hricisákova, 1990; LaSalle, 1960; Lugo *et al.,* 2014; Nápoles Valdés, Juan, 1996; Nápoles Valdés, Juan, 2000; Peng, Lequn and Huang, Lihong, 1994) and classical sources (Lienard, 1928; Van der Pol, 1922; Van der Pol, 1926).

Letting $F(x) = \int_0^x f(s)ds$ we obtain an equivalent system to equation (2):

$$\begin{cases} \dot{x} = y - F(x) \\ \dot{y} = -g(x) \end{cases} \quad \ldots\ldots\ldots\ldots\ldots(3)$$

So, is raise in natural way the study of more general system than (1), therefore we shall study the following system:

$$\begin{cases} \dot{x} = E(x,y) \\ \dot{y} = -p(y)g(x) \end{cases} \quad \ldots\ldots\ldots\ldots\ldots(4)$$

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where E(x, y) = a(x, y) H(x, y) with H(x, y) = $H(\alpha(y)-\beta(y)F(x))$ and functions involved in (4) are continuous real functions in their arguments.

Throughout this paper we will use the following notations (F(x) as above):

$$G(x) = \int_0^x g(s)ds \quad , \quad E^{\alpha}_{a,p}(0,y) = \int_0^x \frac{a(0,s)H(\alpha(s))}{p(s)}ds$$

Our aim in this paper is to obtain sufficient conditions under wich we can ensure the continuability of solutions of (4) in the future. Our results are not only valid for the system (4), but for the systems (1) and (3), and hence, for equation (2).

RESULTS

Theorem 1. Under the following assumptions:

i) E(x, y) is a differentiable function with

$$\sup_{R^2} \frac{\partial E}{\partial x}(x,y) < +\infty \text{ and } \lim_{y\to\pm\infty} \sup E^{\alpha}_{a,p}(y) = \pm\infty$$

ii) p(y) > 0 for all $y \in R$

iii) There exists a positive constant λ such that g(x) F(x) ≥ – λ

$$\int_0^t x(s)g(x(s))ds \ge -\lambda \text{ for all } x \in R \text{ and } \sup_{x\in R}\left|F(x)\right| < +\infty$$

iv) $a \in C^{1,0}(R^2)$ such that a(x, y) > 0 for all x and y, and $\lim_{x\to\pm\infty} \sup a(x,y) < +\infty$ for all y.

then the solutions of system (4) can be defined for all t.

Proof. Suppose on the contrary, there is a solution (x(t), y(t)) of (4) and some T > 0 such that

$$\lim_{t\to T^-} \left(x^2(t)+y^2(t)\right) = +\infty \qquad \text{………… (5)}$$

and define the following Lyapunov's Function:

$$V(x, y) = G(x) + E^{\alpha}_{a,p}(y) \qquad \text{…………….(6)}$$

From definitions of G(x) and $E^{\alpha}_{a,p}(y)$ it follows that V(0, 0) = 0 and V(x, y) > 0 for all x, y ≠ 0.

Differentiating V(x, y) with respect to t along the solutions of system (4) we find:

$$\dot{V}_{(4)}(x,y) = g(x)\dot{x} + \frac{a(0,y)H(0,y)}{p(y)}\dot{y},$$

so we obtain in turn

$$\dot{V}_{(4)}(x,y) = g(x)\,(a(x, y)\,H(x, y) - a(0, y)\,H(0, y))$$

$$\dot{V}_{(4)}(x,y) = g(x)\,(E(x, y) - E(0, y))$$

taking into account the Mean Value Theorem

$$\dot{V}_{(4)}(x,y) = x\, g(x)\, \frac{\partial}{\partial x}E(x_1,y) \qquad \text{…… (7)}$$

where $0 < x_1 < x$.

From i) there is a positive constant M such that $0 \le \frac{\partial E}{\partial x} \le M$ in R^2. Integrating (7) on (0, T) gives

$$V(x, y) - V(x_0, y_0) = \int_0^T g(x(s))\frac{\partial}{\partial x}E(x_1,y(s))x(s)ds \le$$
$$\le \lambda M,$$

with $(x_0, y_0) = (x(0), y(0))$. So, we have

$$V(x, y) \le G(x_0) + E^{\alpha}_{a,p}(y_0) + \lambda M \equiv K$$

Therefore it follows that

$$G(x) + E^{\alpha}_{a,p}(y) \le K \qquad \text{………… (8)}$$

From last assumption in i) and (8) we have

$$\lim_{t\to T^-}\left|y(t)\right| < +\infty \qquad \text{………….. (9)}$$

(5) implies that $\lim_{t\to T^-}\left|x(t)\right| \le +\infty$ but from iii) and iv) it is easy to see that there exists N > 0 such that $\left|\frac{dx(t)}{dt}\right| = \left|E(x,y)\right| \le N$, for all $t \in (0, T)$.

Applying again the Mean Value Theorem we have

$$\left|x(t)-x_0\right| < \left|t-0\right| \left|\frac{dx(\theta)}{dt}\right| \le T\,N \qquad \text{………(10)}$$

for all $t \in (0, T)$ and $0 < \theta < T$. (9) and (10) contradicts the supposition (5). This completes the proof of Theorem.

In similar way, we can prove the following result.

Theorem 2. In addition to i) and iv) suppose that the following assumptions are satisfied:

ii*) p(y) > 0 for all y ∈ R, P(±∞) = ±∞, with $P(y) = \int_0^y \frac{ds}{p(s)}$ and $\limsup_{x\to\pm\infty} G(x) = +\infty$

iii*) there exists a positive constant λ > 0 such that g(x) F(x) > – λ for all x ∈ R and $E^{\alpha}_{a,p}(y) \geq -\lambda$ for all y ∈ R.

Then all the solutions exist in the future.

Remark 1. The above results allow us to obtain new conditions for the boundedness of the solutions of (1) and (2). Thus, for example, for the system (1) the following result is easily obtained from Theorem 1:

Theorem 3. Under the following assumptions:

i) F'(x) > – ∞ and $\lim_{y\to\pm\infty} \alpha(y) = \pm\infty$

ii) there exists a positive constant λ > 0 such that g(x).F(x) > – λ, $\int_0^t x(s)g(x(s))ds \geq -\lambda$ for all x ∈ R and

$\sup_{x\in R}|F(x)| < +\infty$

then all the solutions of system (1) can be defined for all t. The Theorem 3 extends those obtained in (Nápoles Valdés, Juan 1995; Nápoles Valdés, Juan 1996).

Remark 2. When a(x, y) ≡ 1 and β ≡ 1, our results agree with the Theorems 2.1 and 2.2 obtained in (Huang, Li-hong *et al.,* 1999).

Remark 3. Our results are consistent with those reported in the literature for the Lienard's Equation, particularly with those obtained in (Hara *et al.,* 1985; Hasan and Zhu, 2007; Hricisákova, 1990; Lugo *et al.,* 2014; Nápoles Valdés, Juan, 2000; Peng, Lequn and Huang, Lihong, 1994).

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