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RESEARCH ARTICLE

THE NUMBER OF ZEROS OF A POLYNOMIAL IN A DISK

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ABSTRACT

In this paper we give bounds for the number of zeros of a polynomial in a disk when the coefficients of the polynomial are restricted to certain conditions. Our results generalize many known results in this direction and many other new results can also be obtained by a suitable choice of the parameters.
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INTRODUCTION

In 1968, Q. G. Mohammad (1965) considered the problem of finding a bound for the number of zeros of a polynomial inside the unit disk. Under certain conditions on the coefficients of the polynomial, he proved the following result:

Theorem A: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that

$$a_n \ge a_{n-1} \ge ..... \ge a_1 \ge a_0 > 0.$$

Then the number of zeros of P(z) in $|z| \le \frac{1}{2}$ is less than or equal to $1+\frac{1}{\log 2}\log\frac{a_n}{a_0}$.

K. K. Dewan in 1980 (2) generalized Theorem A to polynomials with complex coefficients and proved the following result:

Theorem B: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a complex polynomial of degree n with $a_j=\alpha_j+i\beta_j, j=0,1,2,......,n$, where α_j and β_j are real numbers. If $\alpha_n \ge \alpha_{n-1} \ge ..... \ge \alpha_1 \ge \alpha_0 > 0$, then the number of zeros of P(z) in $|z| \le \frac{1}{2}$ is less than or equal to

$$1+\frac{1}{\log 2}\log\frac{\alpha_n+\sum_{j=0}^{n}|\beta_j|}{|a_0|}.$$

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C. M. Upadhye in 2007 (3) generalized Theorem B by proving the following result:

Theorem C: Let $P(z)=\sum_{j=0}^{n}a_j z^j$ be a complex polynomial of degree n with $a_j=\alpha_j+i\beta_j, j=0,1,2,......,n$, where α_j and β_j are real numbers. If for some $k\geq 1$,

$$k\alpha_n \geq \alpha_{n-1} \geq .....\geq \alpha_1 \geq \alpha_0,$$

then the number of zeros of P(z) in $|z|\leq \delta, 0<\delta<1$ is less than or equal to

$$\frac{1}{\log\frac{1}{\delta}}\log\frac{k(\alpha_n+|\alpha_n|)+|\alpha_0|-\alpha_0+2\sum_{j=0}^{n}|\beta_j|}{|a_0|}.$$

Gulzar in 2012 (4) generalized Theorem C as follows:

Theorem D: Let $P(z)=\sum_{j=0}^{n}a_j z^j$ be a complex polynomial of degree n with $a_j=\alpha_j+i\beta_j, j=0,1,2,......,n$, where α_j and β_j are real numbers. If for some $k\geq 1, 0<\tau\leq 1$,

$$k\alpha_n \geq \alpha_{n-1} \geq .....\geq \alpha_1 \geq \tau\alpha_0,$$

then the number of zeros of P(z) in $|z|\leq \delta, 0<\delta<1$ is less than or equal to

$$\frac{1}{\log\frac{1}{\delta}}\log\frac{k(\alpha_n+|\alpha_n|)+2|\alpha_0|-\tau(\alpha_0+|\alpha_0|)+2\sum_{j=0}^{n}|\beta_j|}{|a_0|}.$$

In 2013, Gulzar (5) proved a more general result as follows:

Theorem E: Let $P(z)=\sum_{j=0}^{n}a_j z^j$ be a complex polynomial of degree n with $a_j=\alpha_j+i\beta_j, j=0,1,2,......,n$, where α_j and β_j are real numbers. If for some $k\geq 1, 0<\tau\leq 1$,

$$k\alpha_n \geq \alpha_{n-1} \geq .....\geq \alpha_1 \geq \tau\alpha_0,$$

then the number of zeros of P(z) in $|z|\leq \frac{R}{c}(R>0, c>1)$ is less than or equal to

$$\frac{1}{\log c}\log\frac{R^{n+1}[k(\alpha_n+|\alpha_n|)+2|\alpha_0|-\tau(\alpha_0+|\alpha_0|)+2\sum_{j=0}^{n}|\beta_j|}{|a_0|} \quad \text{for } R\geq 1$$

and

$$\frac{1}{\log c}\log\frac{|a_0|+R[k(\alpha_n+|\alpha_n|)+|\alpha_0|+|\beta_0|-\tau(\alpha_0+|\alpha_0|)+2\sum_{j=1}^{n}|\beta_j|}{|a_0|} \quad \text{for } R\leq 1.$$

In this paper , we prove the following result which not only contains all the above results as special cases, but also gives many other interesting results for different values of the parameters:

Theorem 1: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a complex polynomial of degree n with $a_j=\alpha_j+i\beta_j, j=0,1,2,......,n$, where α_j and β_j are real numbers. If for some positive integers $\lambda, \mu \le n$ and for some real numbers $0<\rho_1 \le 1, 0<\rho_2 \le 1, k_1 \ge 1, k_2 \ge 1$,

$$k_1^{n-\lambda+1}\alpha_n \ge k_1^{n-\lambda}\alpha_{n-1} \ge k_1^{n-\lambda-1}\alpha_{n-2} \ge ...... \ge k_1^{2}\alpha_{\lambda+1} \ge k_1\alpha_\lambda \ge \alpha_{\lambda-1} \ge ...... \ge \alpha_1 \ge \rho_1\alpha_0,$$

$$k_2^{n-\mu+1}\beta_n \ge k_2^{n-\mu}\beta_{n-1} \ge k_2^{n-\mu-1}\beta_{n-2} \ge ...... \ge k_2^{2}\beta_{\mu+1} \ge k_2\beta_\mu \ge \beta_{\mu-1} \ge ...... \ge \beta_1 \ge \rho_2\beta_0,$$

then the number of zeros of P(z) in $|z| \le \frac{R}{c}(R>0, c>1)$ is less than or equal to

$$\frac{1}{\log c}\log\frac{M}{|a_0|},$$

Where

$$M=|a_n|R^{n+1}+|a_0|+R^n[k_1\alpha_n+k_2\beta_n+(k_1-1)\sum_{j=\lambda}^{n}(\alpha_j+|\alpha_j|)+(k_2-1)\sum_{j=\mu}^{n}(\beta_j+|\beta_j|)$$
$$-\rho_1(\alpha_0+|\alpha_0|)-\rho_2(\beta_0+|\beta_0|)+(|\alpha_0|+|\beta_0|)] \text{ for } R\ge 1,$$

and

$$M=|a_n|R^{n+1}+|a_0|+R[k_1\alpha_n+k_2\beta_n+(k_1-1)\sum_{j=\lambda}^{n}(\alpha_j+|\alpha_j|)+(k_2-1)\sum_{j=\mu}^{n}(\beta_j+|\beta_j|)$$
$$-\rho_1(\alpha_0+|\alpha_0|)-\rho_2(\beta_0+|\beta_0|)+(|\alpha_0|+|\beta_0|)] \text{ for } R\le 1.$$

If the coefficients a_j are real i.e. $\beta_j=0, \forall j$, then we get the following result from Theorem 1 by taking $k_1=k, \rho_1=\rho$:

Corollary 1: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a complex polynomial of degree n. If for some positive integer $\lambda \le n$ and for some real numbers $0<\rho_1 \le 1, 0<\rho_2 \le 1, k \ge 1,$ $k^{n-\lambda+1}a_n \ge k^{n-\lambda}a_{n-1} \ge k^{n-\lambda-1}a_{n-2} \ge ...... \ge k^2 a_{\lambda+1} \ge k a_\lambda \ge a_{\lambda-1} \ge ...... \ge a_1 \ge \rho a_0$, then the number of zeros of P(z) in $|z| \le \frac{R}{c}(R>0, c>1)$ is less than or equal to

$$\frac{1}{\log c}\log\frac{M}{|a_0|},$$

where

$$M=|a_n|R^{n+1}+|a_0|+R^n[ka_n+(k-1)\sum_{j=\lambda}^{n}(a_j+|a_j|)-\rho(a_0+|a_0|)+|a_0|] \text{ for } R\ge 1,$$

and

$$M=|a_n|R^{n+1}+|a_0|+R[ka_n+(k-1)\sum_{j=\lambda}^{n}(a_j+|a_j|)-\rho(a_0+|a_0|)+|a_0|] \text{ for } R\le 1.$$

Taking $\lambda=\mu=n$ in Theorem 1, we get the following result:

Corollary 2: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a complex polynomial of degree n with $a_j=\alpha_j+i\beta_j, j=0,1,2,......,n$, where α_j and β_j are real numbers. If for some real numbers $0<\rho_1 \le 1, 0<\rho_2 \le 1, k_1 \ge 1, k_2 \ge 1$,

$$k_1\alpha_n \geq \alpha_{n-1} \geq ...... \geq \alpha_1 \geq \rho_1\alpha_0$$
$$k_2\beta_n \geq \beta_{n-1} \geq ...... \geq \beta_1 \geq \rho_2\beta_0,$$

then the number of zeros of P(z) in $|z| \leq \frac{R}{c}(R > 0, c > 1)$ is less than or equal to

$$\frac{1}{\log c}\log\frac{M}{|a_0|},$$

where

$$M = |a_n|R^{n+1} + |a_0| + R^n[k_1\alpha_n + k_2\beta_n + (k_1 - 1)(\alpha_n + |\alpha_n|) + (k_2 - 1)(\beta_n + |\beta_n|)$$
$$-\rho_1(\alpha_0 + |\alpha_0|) - \rho_2(\beta_0 + |\beta_0|) + (|\alpha_0| + |\beta_0|)] \text{ for } R \geq 1,$$

and

$$M = |a_n|R^{n+1} + |a_0| + R[k_1\alpha_n + k_2\beta_n + (k_1 - 1)(\alpha_n + |\alpha_n|) + (k_2 - 1)(\beta_n + |\beta_n|)$$
$$-\rho_1(\alpha_0 + |\alpha_0|) - \rho_2(\beta_0 + |\beta_0|) + (|\alpha_0| + |\beta_0|)] \text{ for } R \leq 1.$$

Taking $k_1 = k_2 = k$ in Theorem 1, we get the following result:

Corollary3: Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a complex polynomial of degree n with $a_j = \alpha_j + i\beta_j, j = 0,1,2,......,n$, where α_j and β_j are real numbers. If for some positive integers $\lambda, \mu \leq n$ and for some real numbers $0 < \rho_1 \leq 1, 0 < \rho_2 \leq 1, k \geq 1,$,

$$k^{n-\lambda+1}\alpha_n \geq k^{n-\lambda}\alpha_{n-1} \geq k^{n-\lambda-1}\alpha_{n-2} \geq ......\geq k^2\alpha_{\lambda+1} \geq k\alpha_\lambda \geq \alpha_{\lambda-1} \geq ......\geq \alpha_1 \geq \rho_1\alpha_0,$$
$$k^{n-\mu+1}\beta_n \geq k^{n-\mu}\beta_{n-1} \geq k^{n-\mu-1}\beta_{n-2} \geq ......\geq k^2\beta_{\mu+1} \geq k\beta_\mu \geq \beta_{\mu-1} \geq ......\geq \beta_1 \geq \rho_2\beta_0,$$

then the number of zeros of P(z) in $|z| \leq \frac{R}{c}(R > 0, c > 1)$ is less than or equal to

$$\frac{1}{\log c}\log\frac{M}{|a_0|},$$

where

$$M = |a_n|R^{n+1} + |a_0| + R^n[k(\alpha_n + \beta_n) + (k-1)\sum_{j=\lambda}^{n}(\alpha_j + \beta_j + |\alpha_j| + |\beta_j|)$$
$$-\rho_1(\alpha_0 + |\alpha_0|) - \rho_2(\beta_0 + |\beta_0|) + (|\alpha_0| + |\beta_0|)] \text{ for } R \geq 1, \text{ and}$$

$$M = |a_n|R^{n+1} + |a_0| + R[k(\alpha_n + \beta_n) + (k-1)\sum_{j=\lambda}^{n}(\alpha_j + \beta_j + |\alpha_j| + |\beta_j|)$$
$$-\rho_1(\alpha_0 + |\alpha_0|) - \rho_2(\beta_0 + |\beta_0|) + (|\alpha_0| + |\beta_0|)] \text{ for } R \leq 1.$$

Taking $k_1 = k, \rho_1 = \rho, k_2 = 1$, $\lambda = \mu = n, \rho_2 = 1, \beta_0 > 0$, Cor. 1 gives the following result:

Corollary 4: Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a complex polynomial of degree n with $a_j = \alpha_j + i\beta_j, j = 0,1,2,......,n$, where α_j and β_j are real numbers. If for some real numbers $0 < \rho \leq 1,, k \geq 1,$,

$$k\alpha_n \geq \alpha_{n-1} \geq ...... \geq \alpha_1 \geq \rho\alpha_0$$
$$\beta_n \geq \beta_{n-1} \geq ...... \geq \beta_1 \geq \beta_0 > 0,$$

then the number of zeros of P(z) in $|z| \le \frac{R}{c}(R>0, c>1)$ is less than or equal to

$$\frac{1}{\log c}\log\frac{M}{|a_0|},$$

where

$$M = |a_n|R^{n+1} + |a_0| + R^n[k\alpha_n + \beta_n + (k-1)(\alpha_n + |\alpha_n|) - \rho(\alpha_0 + |\alpha_0|) - \beta_0 + |\alpha_0|] \text{ for } R \ge 1,$$

and

$$M = |a_n|R^{n+1} + |a_0| + R[k\alpha_n + \beta_n + (k-1)(\alpha_n + |\alpha_n|) - \rho(\alpha_0 + |\alpha_0|) - \beta_0 + |\alpha_0|] \text{ for } R \le 1.$$

Many other results can similarly be obtained from the above results by a suitable choice of the parameters.

II. Lemmas

For the proof of Theorem 1, we make use of the following results:

Lemma 1: If f(z) is analytic in $|z| \le R$, but not identically zero, $f(0) \ne 0$ and $f(a_k) = 0,$

k=1,2,……,n,then

$$\frac{1}{2\pi}\int_0^{2\pi}\log|f(\mathrm{Re}^{i\theta}|d\theta - \log|f(0)| = \sum_{k=1}^{n}\log\frac{R}{|a_k|}.$$

Lemma 1 is the famous Jensen's Theorem(see page 208 of (1)).

Lemma 2: If f(z) is analytic $f(0) \ne 0, |f(z)| \le M$ in $|z| \le R$, then the number of zeros of f(z) in $|z| \le \frac{R}{c}, c>1$ does not exceed

$$\frac{1}{\log c}\log\frac{M}{|f(0)|}.$$

Lemma 2 is a simple consequence of Lemma 1.

III. Proofs of Theorems

Proof of Theorem 1: Consider the polynomial

$$\begin{aligned}
F(z) &= (1-z)P(z) \\
&= (1-z)(a_n z^n + a_{n-1}z^{n-1} + \ldots\ldots + a_1 z + a_0) \\
&= -a_n z^{n+1} + (a_n - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} + \ldots\ldots + (a_1 - a_0)z + a_0 \\
&= -a_n z^{n+1} + a_0 + (\alpha_n - \alpha_{n-1})z^n + (\alpha_{n-1} - \alpha_{n-2})z^{n-1} + \ldots\ldots + (\alpha_1 - \alpha_0)z \\
&\qquad + i\{(\beta_n - \beta_{n-1})z^n + (\beta_{n-1} - \beta_{n-2})z^{n-1} + \ldots\ldots + (\beta_1 - \beta_0)z\} \\
&= -a_n z^{n+1} + a_0 + (k_1\alpha_n - \alpha_{n-1})z^n + (k_1\alpha_{n-1} - \alpha_{n-2})z^{n-1} + \ldots\ldots + (k_1\alpha_{\lambda+1} - \alpha_\lambda)z^{\lambda+1} \\
&\qquad + (k_1\alpha_\lambda - \alpha_{\lambda-1})z^\lambda + (\alpha_{\lambda-1} - \alpha_{\lambda-2})z^{\lambda-1} + \ldots\ldots + (\alpha_1 - \rho_1\alpha_0)z + (\rho_1\alpha_0 - \alpha_0)z \\
&\qquad - (k_1 - 1)(\alpha_n z^n + \alpha_{n-1}z^{n-1} + \ldots\ldots + \alpha_\lambda z^\lambda) + i\{(k_2\beta_n - \beta_{n-1})z^n + (k_2\beta_{n-1} - \beta_{n-2})z^{n-1} \\
&\qquad + \ldots\ldots + (k_2\beta_{\mu+1} - \beta_\mu)z^{\mu+1} + (k_2\beta_\mu - \beta_\mu)z^\mu + (\beta_{\mu-1} - \beta_{\mu-2})z^{\mu-1} + \ldots\ldots \\
&\qquad + (\beta_1 - \rho_2\beta_0)z + (\rho_2\beta_0 - \beta_0)z - (k_2 - 1)(\beta_n z^n + \beta_{n-1}z^{n-1} + \ldots\ldots + \beta_\mu z^\mu)\}.
\end{aligned}$$

For $|z| \le R$, we have, by using the hypothesis,

$$
\begin{aligned}
|F(z)| \le\ & |a_n|R^{n+1} + |a_0| + (k_1\alpha_n - \alpha_{n-1})R^n + (k_1\alpha_{n-1} - \alpha_{n-2})R^{n-1} + ...+ (k_1\alpha_{\lambda+1} - \alpha_\lambda)R^{\lambda+1} \\
& + (k_1\alpha_\lambda - \alpha_{\lambda-1})R^\lambda + (\alpha_{\lambda-1} - \alpha_{\lambda-2})R^{\lambda-1} + ...+ (\alpha_1 - \rho_1\alpha_0)R + (\rho_1\alpha_0 - \alpha_0)R \\
& + (k_1 - 1)(|\alpha_n|R^n + |\alpha_{n-1}|R^{n-1} + ...+ |\alpha_\lambda|R^\lambda) + (k_2\beta_n - \beta_{n-1})R^n \\
& + (k_2\beta_{n-1} - \beta_{n-2})R^{n-1} + ...+ (k_2\beta_{\mu+1} - \beta_\mu)R^{\mu+1} + (k_2\beta_\mu - \beta_\mu)R^\mu \\
& + (\beta_{\mu-1} - \beta_{\mu-2})R^{\mu-1} + ...+ (\beta_1 - \rho_2\beta_0)R + (\rho_2\beta_0 - \beta_0)R \\
& + (k_2 - 1)(|\beta_n|R^n + |\beta_{n-1}|R^{n-1} + ... + |\beta_\mu|R^\mu)\}
\end{aligned}
$$

For $R \ge 1$, we have

$$
\begin{aligned}
|F(z)| \le\ & |a_n|R^{n+1} + |a_0| + R^n[(k_1\alpha_n - \alpha_{n-1}) + (k_1\alpha_{n-1} - \alpha_{n-2}) + ...+ (k_1\alpha_{\lambda+1} - \alpha_\lambda) \\
& + (k_1\alpha_\lambda - \alpha_{\lambda-1}) + (\alpha_{\lambda-1} - \alpha_{\lambda-2}) + ...+ (\alpha_1 - \rho_1\alpha_0) + (1-\rho_1)|\alpha_0|) \\
& + (k_1 - 1)(|\alpha_n| + |\alpha_{n-1}| + ... + |\alpha_\lambda|) + (k_2\beta_n - \beta_{n-1}) \\
& + (k_2\beta_{n-1} - \beta_{n-2}) + ... + (k_2\beta_{\mu+1} - \beta_\mu) + (k_2\beta_\mu - \beta_\mu) \\
& + (\beta_{\mu-1} - \beta_{\mu-2}) + ...+ (\beta_1 - \rho_2\beta_0) + (1-\rho_2)|\beta_0| \\
& + (k_2 - 1)(|\beta_n| + |\beta_{n-1}| + ... + |\beta_\mu|)] \\
=\ & |a_n|R^{n+1} + |a_0| + R^n[(k_1\alpha_n + k_2\beta_n) + (k_1 - 1)\sum_{j=\lambda}^{n}(\alpha_j + |\alpha_j|) + (k_2 - 1)\sum_{j=\lambda}^{n}(\beta_j + |\beta_j|) \\
& - \rho_1(\alpha_0 + |\alpha_0|) - \rho_2(\beta_0 + |\beta_0|) + (|\alpha_0| + |\beta_0|)]
\end{aligned}
$$

and for $R \le 1$, we have

$$
\begin{aligned}
|F(z)| \le\ & |a_n|R^{n+1} + |a_0| + R[(k_1\alpha_n + k_2\beta_n) + (k_1 - 1)\sum_{j=\lambda}^{n}(\alpha_j + |\alpha_j|) + (k_2 - 1)\sum_{j=\lambda}^{n}(\beta_j + |\beta_j|) \\
& - \rho_1(\alpha_0 + |\alpha_0|) - \rho_2(\beta_0 + |\beta_0|) + (|\alpha_0| + |\beta_0|)].
\end{aligned}
$$

Since F(z) is analytic for $|z| \le R$ and $F(0) = a_0$, by Lemma 2, it follows that the number of zeros of F(z) in $|z| \le \frac{R}{c}(R > 0, c > 1)$ is less than or equal to

$$
\frac{1}{\log c}\log\frac{M}{|a_0|},
$$

where

$$
\begin{aligned}
M =\ & |a_n|R^{n+1} + |a_0| + R^n[k_1\alpha_n + k_2\beta_n + (k_1 - 1)\sum_{j=\lambda}^{n}(\alpha_j + |\alpha_j|) + (k_2 - 1)\sum_{j=\mu}^{n}(\beta_j + |\beta_j|) \\
& - \rho_1(\alpha_0 + |\alpha_0|) - \rho_2(\beta_0 + |\beta_0|) + (|\alpha_0| + |\beta_0|)] \text{ for } R \ge 1,
\end{aligned}
$$

and

$$
\begin{aligned}
M =\ & |a_n|R^{n+1} + |a_0| + R[k_1\alpha_n + k_2\beta_n + (k_1 - 1)\sum_{j=\lambda}^{n}(\alpha_j + |\alpha_j|) + (k_2 - 1)\sum_{j=\mu}^{n}(\beta_j + |\beta_j|) \\
& - \rho_1(\alpha_0 + |\alpha_0|) - \rho_2(\beta_0 + |\beta_0|) + (|\alpha_0| + |\beta_0|)] \quad \text{for } R \le 1.
\end{aligned}
$$

Since the zeros of P(z) are also the zeros of F(z), it follows that the number of zeros of P(z) in $|z| \leq \frac{R}{c}(R > 0, c > 1)$ is less than or equal to

$$\frac{1}{\log c}\log\frac{M}{|a_0|},$$

where

$$M = |a_n|R^{n+1} + |a_0| + R^n[k_1\alpha_n + k_2\beta_n + (k_1 - 1)\sum_{j=\lambda}^{n}(\alpha_j + |\alpha_j|) + (k_2 - 1)\sum_{j=\mu}^{n}(\beta_j + |\beta_j|)$$
$$-\rho_1(\alpha_0 + |\alpha_0|) - \rho_2(\beta_0 + |\beta_0|) + (|\alpha_0| + |\beta_0|)] \text{ for } R \geq 1,$$

and

$$M = |a_n|R^{n+1} + |a_0| + R[k_1\alpha_n + k_2\beta_n + (k_1 - 1)\sum_{j=\lambda}^{n}(\alpha_j + |\alpha_j|) + (k_2 - 1)\sum_{j=\mu}^{n}(\beta_j + |\beta_j|)$$
$$-\rho_1(\alpha_0 + |\alpha_0|) - \rho_2(\beta_0 + |\beta_0|) + (|\alpha_0| + |\beta_0|)] \text{ for } R \leq 1$$

and the proof of Theorem 1 is complete.

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