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RESEARCH ARTICLE

ON AN INVERSE PROBLEM FOR AN ELLIPTIC EQUATION WITH NON-LINEAR BOUNDARY CONDITIONS

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ABSTRACT

This study explores the well-posedness of an inverse problem for an elliptic equation with non-linear boundary conditions. The problem is analyzed in a bounded domain under Neumann boundary conditions with additional integral information. A theorem on the uniqueness and stability of the solution is presented and proved.

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INTRODUCTION

Inverse problems for elliptic equations often arise in various scientific and engineering applications such as geophysics, medical imaging, and material sciences. These problems are typically ill-posed, meaning that solutions may not exist, be unique, or depend continuously on the data. This paper extends previous studies by considering an inverse problem for an elliptic equation with non-linear boundary conditions and integral additional information, which are more reflective of real-world conditions.

Let $D \subset R^n$ be a convex, bounded domain with a sufficiently smooth boundary ∂D. The goal is to determine a pair of functions {f(x), u(x, y)} from the given conditions:

Au = f(x) g(u) in D,

$\frac{\partial u}{\partial n} +$ ku = q(x, y) on ∂D,

$\int_{\gamma_1(x)}^{\gamma_2(x)} u(x,y)dy = h(x)\ in$ D' .

The focus will be on establishing the conditions under which the solution is unique and stable, thereby ensuring the well-posedness of the problem.

2. Assumptions and Definitions: To proceed with the analysis, we make the following assumptions about the input data:

- The function g(u) is Lipschitz continuous.
- The domain D has a $C^{2+\alpha}$ boundary, where $0 < \alpha < 1$.
- The functions h(x) and q(x, y) are sufficiently smooth and belong to $C^{2+\alpha}$ spaces.
- The functions $\gamma_1(x)$ and $\gamma_2(x)$ are well-defined and satisfy $\gamma_1(x) < \gamma_2(x)$

We define the solution of the inverse problem as follows:

Definition 2.1: A pair of functions { f(x), u(x, y) is said to be a solution of the inverse problem if $f \in C(D)$ and $u \in C^2(D)$, and they satisfy the conditions outlined in the problem statement.

3. Uniqueness and Stability Theorem: We now state and prove the main theorem regarding the uniqueness and stability of the solution.

Theorem 3.1: Let the functions g(u), q(x, y), h(x) satisfy the given conditions. Then the solution { f(x), u(x, y)} of the inverse problem is unique and the following stability estimate holds:

$$\| f_1-f_2\|_0 + \| u_1- u_2 \|_0 \leq C (\|g_1 - g_2 \|_0 + \|q_1 - q_2\|_0 + \|h_1 - h_2 \|_0)$$

where C is a constant dependent on the data of the problem.

Proof of Theorem 3.1: To prove the theorem, we begin by integrating the elliptic equation with respect to the variable y over the interval $[\gamma_1(x), \gamma_2(x)]$. By applying the given conditions and assumptions, we obtain a system of equations for the differences between two solutions, $\{f_1(x), u_1(x, y)\}$ and $\{f_2(x), u_2(x, y)\}$.

Let $z(x, y) = u_1(x, y) - u_2(x, y)$ and $X(x) = f_1(x) - f_2(x)$. Then, the difference z(x, y) satisfies the following system:

$$Az = X(x)\, g_1(u_1) + F(x, u_1, u_2) \text{ in } D,$$

$$\frac{\partial z}{\partial n} + kz = q_1(x, y) - q_2(x, y) \text{ on}\} \ \partial D,$$

$$\int_{\gamma_1(x)}^{\gamma_2(x)} z(x, y)\,dy = h_1(x) - h_2(x) \text{ in } D'.$$

Using the properties of the Green's function and the given conditions, we derive the stability estimate for z(x, y) and X(x). The detailed steps of the proof involve applying the maximum principle for elliptic equations and estimating the Green's function. Finally, we obtain the desired stability estimate, proving the uniqueness of the solution.

4. Numerical Methods and Applications: The theoretical results presented in this study have significant implications for practical applications. In many real-world scenarios, solving inverse problems with non-linear boundary conditions is crucial for accurate modeling and analysis. Numerical methods such as the finite element method (FEM) and the boundary element method (BEM) can be employed to approximate solutions of the inverse problem. In this section, we outline a numerical algorithm for solving the inverse problem using FEM. The algorithm involves discretizing the domain D into a finite number of elements and applying the weak formulation of the elliptic equation. By iteratively updating the solution based on the given conditions and minimizing the residual error, we can obtain an approximate solution that satisfies the stability estimate.

5. Case Study: Heat Conduction in Non-Homogeneous Materials: To demonstrate the practical application of the proposed method, we consider a case study of heat conduction in non-homogeneous materials. The goal is to determine the spatial distribution of thermal conductivity f(x) and the temperature field u(x, y) within a bounded domain D with non-linear boundary conditions. The heat conduction equation is modeled as an elliptic equation with non-linear boundary conditions, and the inverse problem involves determining the thermal conductivity function based on temperature measurements at the boundary. The numerical algorithm presented in the previous section is applied to solve this inverse problem, and the results are analyzed to validate the theoretical findings.

6. CONCLUSION

This study presents a comprehensive analysis of an inverse problem for an elliptic equation with non-linear boundary conditions. The uniqueness and stability of the solution are established under given assumptions, and a numerical algorithm is proposed for practical applications. The case study on heat conduction in non-homogeneous materials demonstrates the effectiveness of the proposed method.

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