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RESEARCH ARTICLE

NUMERICAL ANALYSIS OF MODIFIED GINGERBREADMAN MAP AND ITSSYNCHRONIZATION IN A MULTIPLEX NETWORK

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ABSTRACT

Gingerbreadman maps are two-dimensional mathematical model in dynamical systems that depicts chaotic behavior and has the ability to generate complex patterns when iterated. The numerical analysis of a modified Gingerbreadman map and its synchronization in a multiplex network is presented in this paper. A thorough numerical analysis of this model reveals a rich transition between periodic, quasi-periodic, and chaotic regimes. The domain of synchronization is probed by designing a three-layer network of 200 chaotic modified Gingerbreadman maps. The numerical analysis of this domain through interlayer coupling force reveals the layers to be desynchronous or synchronous state. The inherent unpredictability of this model renders it a promising candidate for cryptographic applications, particularly in the generation of robust pseudo-random numbers. These numbers exhibit exceptional resistance to attack, thereby enhancing the security of scientific data, as well as facilitating secure communication, leading to economic development.

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INTRODUCTION

Nowadays, nonlinear dynamic systems are fundamental to the study of complex phenomena (Misbah, *et al.* 2011). They are found in various fields such as physics (Nana et al., 2006), engineering (Roy *et al.,* 2011), biology (Bidesh et al., 2015), technology, and many others. The study of their dynamic behavior, which often produces rich and unpredictable dynamics such as periodic, quasi-periodic, and chaotic regimes, is of great interest to researchers (Murphy et al., 2010). After Lorenz's investigation (Lorenz *et al.*,1963) , the study of chaotic systems has exploded into discrete systems called chaotic maps. Among the most studied discrete chaotic maps are the logistic map (Kalsouabe *et al.,* 2023), the Henon map (Hénon, *et al.,* 1976*),* the Duffing map (Deiva Sundari *et al.,* 2025), and the Gingerbreadman map (Sharma *et al.,* 2023). These maps can lead to more complex attractors, showing sensitivity to initial conditions and robust chaotic regimes. Faced with the growing increase in complex systems, the synchronization of chaotic systems has become a fundamental phenomenon in the study of coupled map dynamics (Mahamat Hasane *et al.*,2025). The synchronization of chaos in complex networks is the subject of intense research, as evidenced by the plethora of papers published on these topics (Mahamat Hasane *et al.,* 2025; Shepelev *et al.* 2021). However, scientists have recently turned their attention toward the synchronization of single-layer and multi-layer networks (Woafo *et al*., 2000; Chen *et al.,* 2025; Omelchenko *et al.,* 2011). Multi-layer networks have become essential for describing interactions spread across several levels (Bródka *et al.,* 2020; Hammoud *et al*., 2020). Unlike a single-layer network, which represents connections between similar nodes, multi-layer networks can represent nodes belonging to different classes or layers. In multilayer networks, synchronization can be intralayer, i.e., within the same layer, and also interlayer between different layers, thus opening up very rich dynamics (Boccaletti *et al.,* 2000; De Domenico *et al.,* 2015). Recent work has shown that inter-layer synchronization is strongly influenced by the inter-layer coupling strength and also by the geometric configuration of the layers (Zhou *et al*., 2020; Zhou *et al*. 2019; Zhou *et al.,* 2022). In this context, the modified Gingerbreadman card is a wise choice and proves particularly effective. This paper aims to analyze the dynamic behavior of the modified one-dimensional Gingerbreadman card and then conduct an in-depth synchronization of a three-layer network consisting of several coupled chaotic modified Gingerbreadman maps. This approach presents an original contribution demonstrating the ability of a discrete, simple model to produce complex dynamics in a multi-layer architecture, thus opening up prospects for cryptographic applications. The remaining paper is organized as follows. Section 2 deals with the numerical analysis of chaotic modified Gingerbreadman map.Section 3 presents synchronization in a multiplex network of coupled modified Gingerbreadman maps.The conclusion is addressed in section 4.

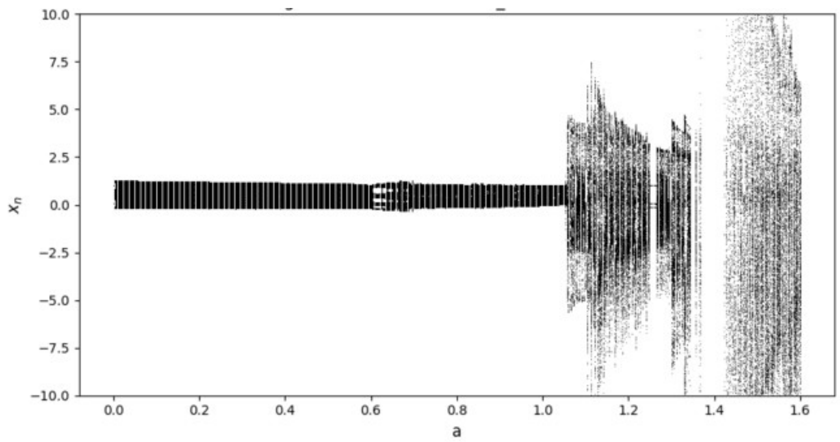
Numerical analysis of the modified Gingerbreadman map

The modified Gingerbreadman map is described (Sharma *et al.,* 2023) **by**:

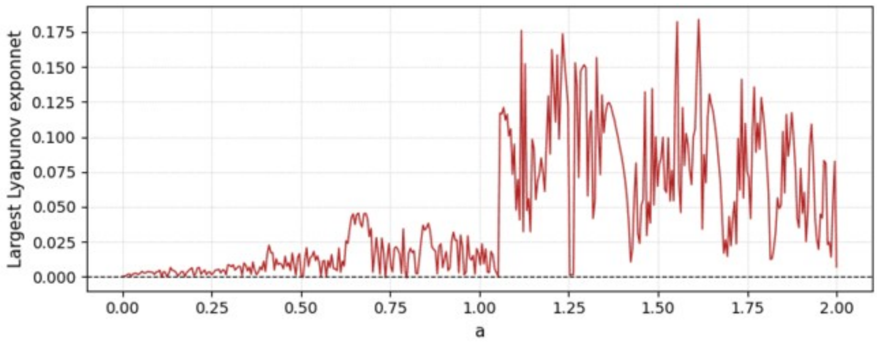
$$x_{n+1} = y_n, \tag{1a}$$

$$y_{n+1} = 1 - y_n + \left[a\cos(2\pi b)\right]|x_n| \ , \tag{1b}$$

Where n is aninteger number, a , b are positive constants and x_n, y_n, respectively denote the n[th] states. Sharma and Joshihave been proposed amap (1) by introducing a cosine term in Gingerbreadman map to observe various eye-catching attractors [10], and numerical analysis shows a correlation between the beauty of arts and science for particular values of a = 2, and 0.125 ≤ b ≤ 0.4. Figure 1 depicts the bifurcation diagrams of x_n and its matching Lyapunov exponent versus for b= 0.45.



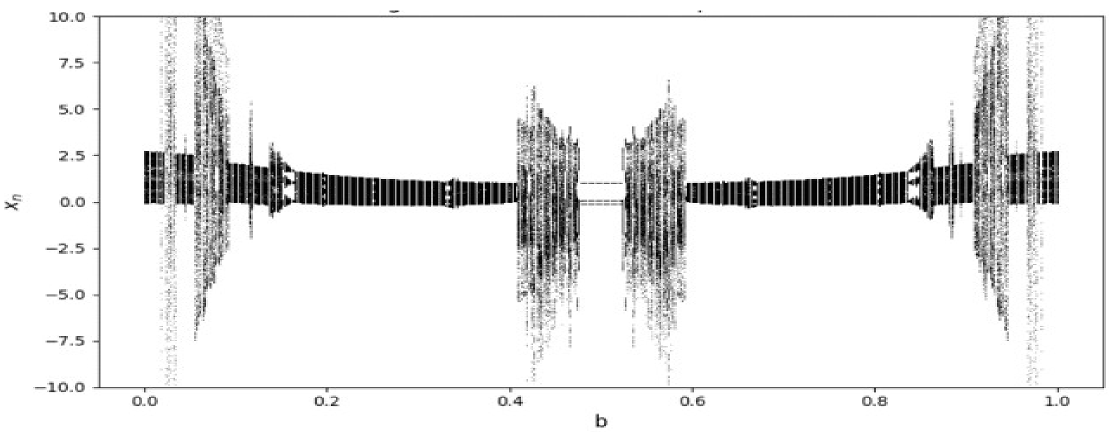
A



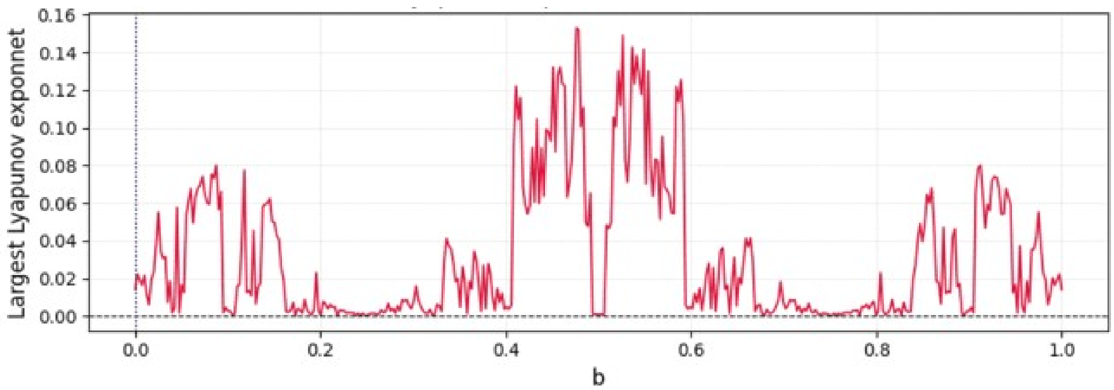
B

Figure 1. Bifurcation diagram of x_n (a) and its matching Lyapunov exponent (b) of map (1) versus parameter a for b= 0.45

In Figure 1 (a) depicts periodic or quasi-periodic behavior for a specific range of values: 0 ≤ a ≤ 1.05, and chaotic behavior for 1.05 ≤ a ≤ 1.6. As demonstrated in Figure 1(b), the maximum Lyapunov exponents exhibit a high degree of correlation with the behavior of the bifurcation diagram in Figure 1 (a). The bifurcation diagrams of x_n and the Lyapunov exponent versus parameter b for a= 1.2 is display in Figure 2.



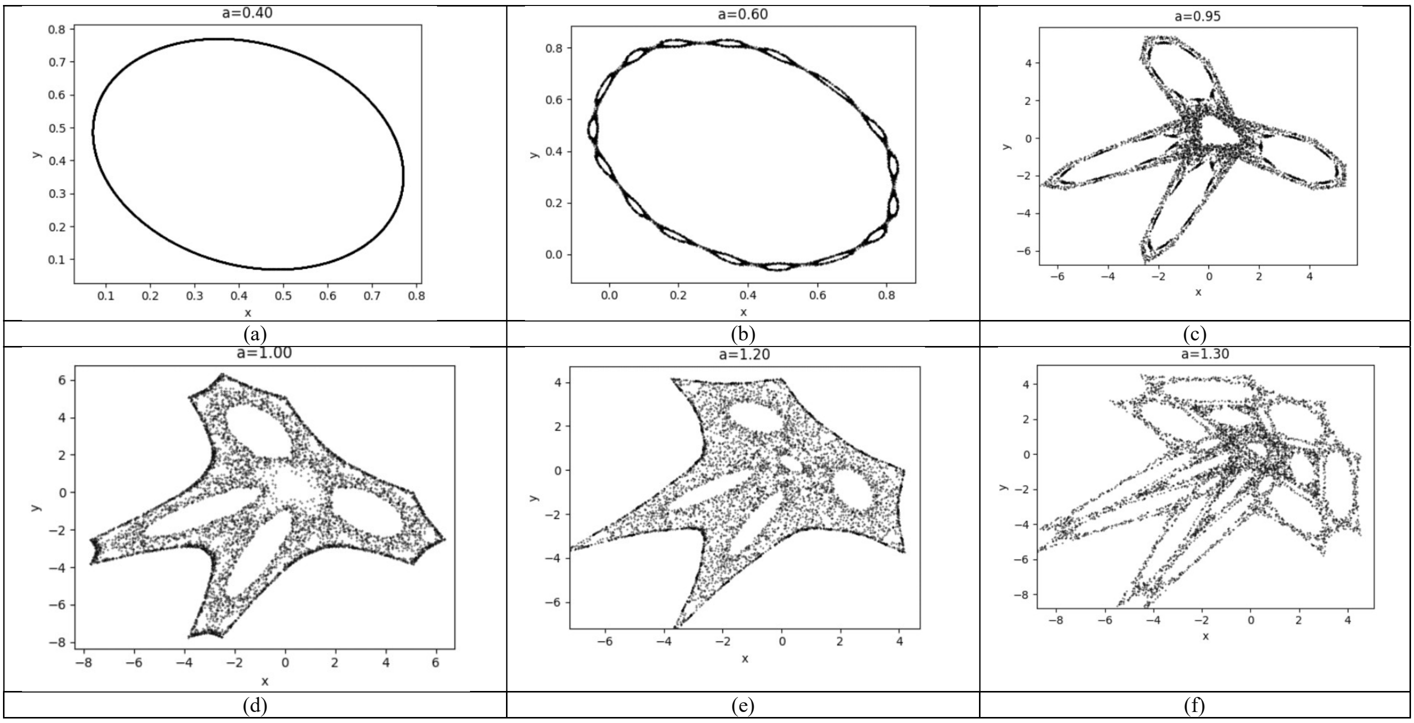
A



B

Figure 2. Bifurcation diagram of x_n (a) and Lyapunov exponent (b) of map (1) versus parameter b for b= 1.2

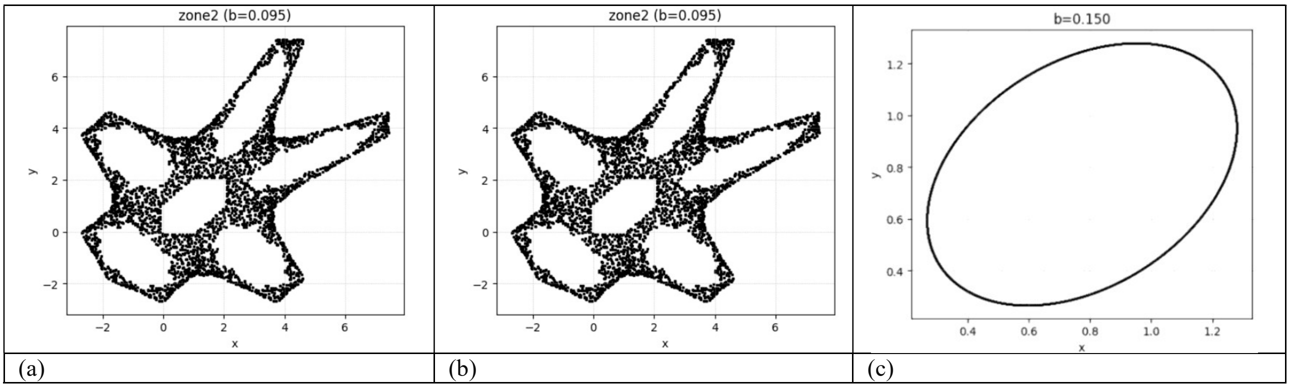
In Figure 2 (a), uncovers chaotic behavior in six zones for some range of values of b:: 0 ≤ b ≤ 0.06; 0.07≤ b ≤ 0.12; 0.28 ≤ b ≤ 0.33; 0.40 ≤ b ≤ 0.67; 0.84 ≤ b ≤ 0.90, and 0.94 ≤ b ≤ 1. Periodic or quasi-periodic behavior is also observed. As illustrated in Figure 2(b), the maximum Lyapunov exponents exhibit a high degree of correlation with the behaviour of the bifurcation diagram in Figure 2(a). The corresponding.



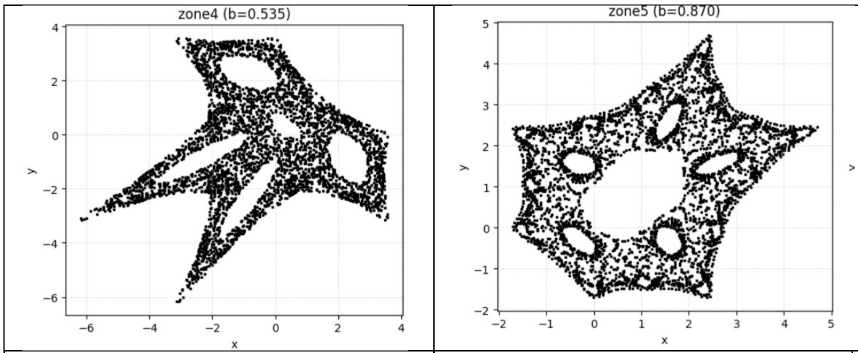
(a) (b) (c) (d) (e) (f)

Figure 3: Phase portraits of map (1) for a given value of b **= 0.45, and** for different a values **(a) a = 0.4, (b) a = 0.6,(c) a = 0.95,(d) a = 1,(e) a = 1.20,and (f) a = 1.30.**

Figure 3(a) clearly proves quasi-periodic behavior for a = 0.40, and periodic behavior for b = 0.60 as shown in Figure 3(b). Figures 3(d), (e), and (f) exhibit chaotic behavior for b = 0.95, b = 1, b = 1.2, and b = 1.3. Figure 4 shows the phase portraits of the different dynamical behaviors of Figure 2.



(a) (b) (c)



(d) (e)

Figure 4. Phase portraits of map (1) for given values of fixe a= 1.2, , and for different b values **(a) b = 0.03, (b) b = 0.095,(c) b = 0.150,(d) b = 0.535 and(e) a = 0.870.**

Figures 5 (a) and (c), depicts two types of geometries of periodic attractor for particular values of b = 0.030 and b = 0.15. For b = 0.095, b = 0.535, and b = 0.870, Figures 5(b), (d), and (e) exhibit chaotic behaviour

Synchronization in three layers network of coupled chaotic modified Gingerbreadman maps: This section presents the synchronization state in 3-layer network where each layer contains 200 modified Gingerbreadman cards coupled together by an intra-layer coupling force. The 3 layers interact with each other through the inter-layer coupling force. The 3-layer network is given by:

where $x_i$and $y_i$are the actual dynamic variables, i = 1, 2, …. N is the number of oscillators in the layer, the layer index is given by m [1, ..., 3], n is the discrete time, is the intra-layer coupling force, σ_{ml} is the inter-layer coupling force, P_m is the non-local intra-layer coupling related to the dimensionless coupling radius r=P/N. The functions $f_x(x,y) = y(n)$ *and* $f_y(x,y) = 1 - y_n + \left[a\cos(2\pi b)\right]|x_n|$ are modified one-dimensional discrete Gingerbreadman maps. The values of the system parameters *a* and *b* are chosen to obtain chaotic behavior. Inter-layer connections exist only between the same nodes per layer, as shown in Figure 5.

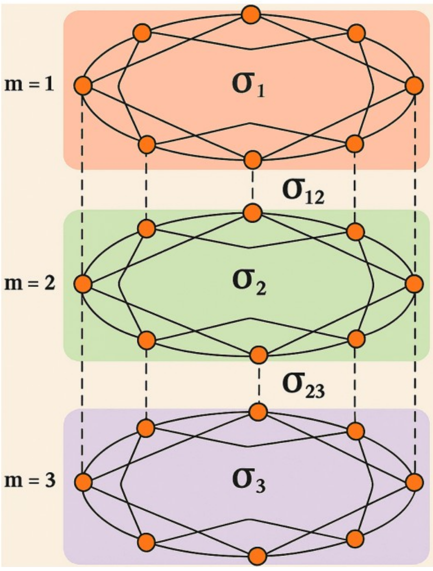


Figure 5. Three-layer network of 200 chaotic modified Gingerbreadman maps

Figure 6 shows a multiplex network consisting of three layers, each containing 200 modified Gingerbreadman cards. Each layer consists of a set of chaotic cards coupled by two types of connections: intra-layer coupling (coupling within each layer) and inter-layer coupling (coupling between different layers). This multilayer architecture makes it possible to study collective dynamic behavior within layers and through interactions between layers, and will also be used to circulate information from one layer to another when all layers are synchronized with each other, as shown by (Mahamat Hasane *et al.,* 2025).To study the synchronization between two layers *l* and *m* in Figure 5, the inter-layer synchronization error *Eml* will be used. It is defined by equation 3:

with || − || as the Euclidean norm and T as the number of time steps. For synchronization, we must measure the synchronization error between the first and third layers E13 and between the first and second layers E12. Thus, global synchronization occurs when E13 = 0 and E12 = 0, and partial synchronization occurs when E13 or E12 differs from 0. However, when E13 and E12 differ from 0, this is referred to as desynchronization. Figure 6 shows the synchronization error between layers as a function of the interlayer coupling strength.

In Figure 6, the synchronization errors (E12 and E13) are zero, the three layers are synchronized with each other, and if they are not zero, then there is no synchronization. To confirm the measurements in Figure 6, two values of σ_{ml} (one value when *Eml* is zero and one when *Eml* is not zero) are taken to plot the spatial traces *yi* as a function of *i*, as displayed in Figure 7.

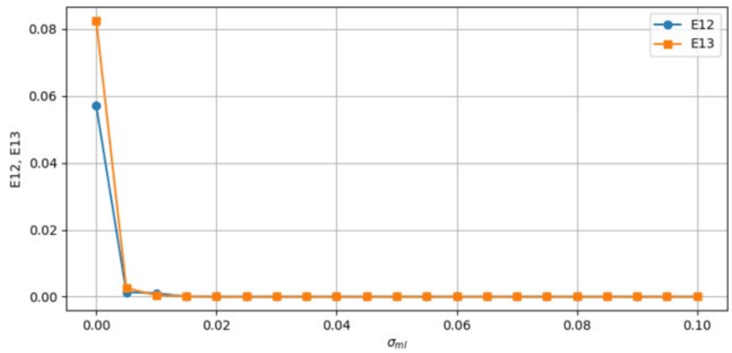


Figure 6. Synchronization error between the first and second layers (E12) and between the first and third layers (E13) as a function of the interlayer coupling strength for a= 1.20 and b= 0.45.

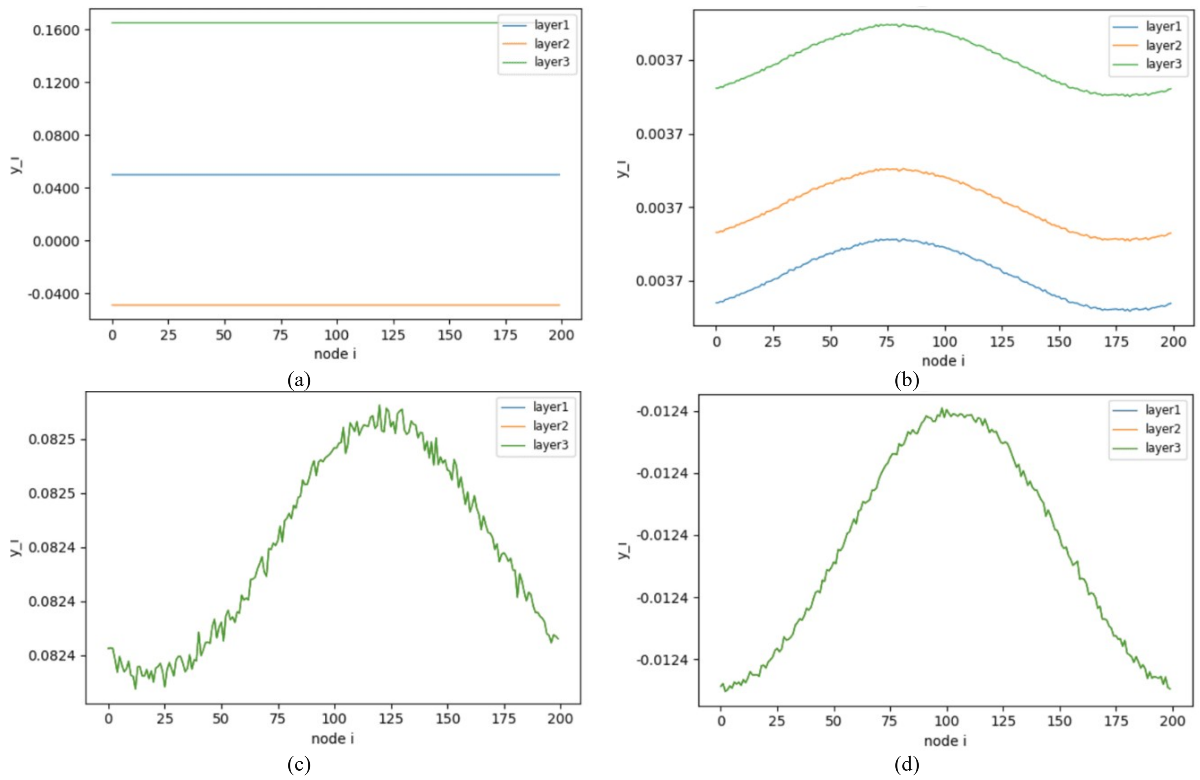


Figure 7. Different spatial profiles depending on the nodes for N= 200, Pm= 70, a= 1.20, b= 0.45, σ_m = 0.03, with initial conditions are chosen randomly in [−2, 2]: a) σ_{ml} = 0, b) σ_{ml} = 0.05, c) σ_{ml} = 0.02, and d) σ_{ml} = 0.05

For a value of σ_{ml} = 0 in Figure 7 (a), when there is no interlayer coupling, the spatial traces show that each layer (yi1, yi2, and yi3) is seen as a horizontal line. The horizontal lines show a special homogeneity caused by non-local diffusion. For a small value of the interlayer coupling force σ_{ml} = 0.005, as shown in Figure 7 (b), the layers produce weak spatial modulation and are not completely synchronized. This is visible in the dynamic behavior of each layer (yi1, yi2, and yi3). Each layer has a different amplitude. The modified discrete Gingerbreadman maps of each layer follow their own dynamics, indicating desynchronization despite the weak coupling force applied. However, when the coupling force values are chosen in the range where E12 and E13= 0 (σ_{ml} = 0.02 and 0.05), the modified discrete Gingerbreadman maps of these three layers produce the same dynamic behavior showing complete synchronization, as shown in Figures 7(c) and (d).

CONCLUSION

This article was devoted to the numerical analysis of modified Gingerbreadman map and its synchronization in threelayers network. The numerical study of the modified Gingerbreadman map demonstrateda wealth of complex phenomena showing a transition between periodic, quasi-periodic, and chaotic regimes. Next, the values for which the parameters leading to chaos were used to couple 200 modified Gingerbreadman cells in a 3-layer network through multiplex synchronization. In addition, numerical analysis revealed complete synchronization for certain values of the interlayer coupling forces. This result highlights the dynamics of maps that were coupled together but remain synchronized despite their chaotic nature. This work thus opens up prospects for cryptographic applications using the synchronization of coupled chaotic maps.

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