



RESEARCH ARTICLE

EXISTENCE OF NASH EQUILIBRIUM IN GENERALIZED B-METRIC SPACES VIA MULTI-VALUED FIXED-POINT THEOREMS

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ABSTRACT

In this paper, the existence problem for Nash equilibrium is restated as a fixed point problem for a multivalued mapping in a generalized b-metric strategy profile space with metric contractivity rather than the traditional topological properties of compactness, convexity, and continuity. The strategy space is endowed with a complete b-metric with constant $s \geq 1$, and the set-valued best response mapping Bis closed and bounded-valued with Pompeiu-Hausdorff contraction property $H_d(B(x), B(y)) \leq \lambda d(x, y)$. The central result establishes the compatibility threshold $s\sqrt{\lambda} < 1$, at which Bis has a unique fixed point identical to a Nash equilibrium. The bound provides the trade-off between the level of contraction and the magnification caused by relaxing the triangle inequality, reverting to the well-known condition $\lambda < 1$ when $s = 1$. Based on this existence result, the article provides a geometric rate of convergence for best response iterations, a perturbation stability result for best response correspondences that are uniformly close, and a Tykhonov-type well-posedness result with extra assumptions. A two-person game with true set-valued best responses is solved analytically as an example for validating the assumptions, with some sensitivity analysis of the threshold $s\sqrt{\lambda} < 1$. Contribution The contribution is methodological in that it finds sufficient conditions for equilibrium existence using metric completeness and contractivity without depending mainly on convexity and upper semicontinuity, and incorporates best response sensitivity explicitly.

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INTRODUCTION

Nash equilibrium is defined by a strategy profile such that the strategy of every player is optimal taking into account the strategies of other players. Equilibrium existence can be stated as a fixed-point problem for the best-response mapping. Such an approach was first proposed by means of Brouwer's fixed-point theorem for continuous single-valued mappings on compact convex sets (1) and later extended by Kakutani to upper semicontinuous convex-valued correspondences (2); Nash employed this generalization for finite non-cooperative games (3,4). The extension to locally convex topological vector spaces is due to Fan and Glicksberg (5,6), and Debreu extended it to problems where the set of feasible strategies depends on strategies (7). These approaches rely mainly on compactness, convexity, and continuity or upper semicontinuity. In the case of strategy spaces having a natural metric structure, another possibility is to substitute the above topological conditions with completeness and contractivity of the best-response mapping. The b-metric framework of Bakhtin and Czerwik (8,9) relaxes the triangle inequality to $d(x, z) \leq s[d(x, y) + d(y, z)]$, $s \geq 1$, while maintaining an underlying basis for Banach contraction mapping type results (10). The extended b-metric setting also allows a point dependent coefficient (11). Given that the best response correspondence could have multiple values, the appropriate fixed point theorem is the multi-valued contraction mapping theorem first introduced by Nadler (12) and subsequently generalized in the b-metric and quasi-contraction cases by Aydi, Bota, Karapinar and collaborators (13). This article combines both approaches through considering the joint best response correspondence in a complete b-metric space such that Pompeiu-Hausdorff contractivity implies a fixed point and Nash equilibrium. It is also possible to prove directly the relationship between equilibrium concepts and fixed point theorems. In particular, Heikkilä and Reffett proved set-valued fixed point theorems for Nash equilibria (14) and Yu, Wang, and Yang provided equivalent theorems relating Nash equilibrium and Brouwer- and Kakutani-type fixed point theorems (15). These works lend further support to the current methodological approach of finding a corresponding equilibrium-generating operator whose fixed points represent the equilibria. However, there has been little relationship between the work done on topological equilibrium existence and multi-valued contractions in generalized metric spaces. This paper bridges

this gap by defining the product best response correspondence as a Hausdorff contraction in a generalized b-metric space of strategy profiles. Specific goals are: (i) to endow the space of strategy profiles with a full-fledged generalized b-metric space structure; (ii) to express the collective best response mapping as a multi-valued mapping under a Pompeiu–Hausdorff contraction property; (iii) to find conditions under which it has a fixed point using the multi-valued b-metric theorem; (iv) to find the corresponding Nash equilibrium existence result; (v) to get the necessary condition $s\sqrt{\lambda} < 1$; (vi) to formulate an illustrative two-person game featuring set-valued best responses that satisfy the necessary hypotheses. The existence of equilibrium is addressed independently of uniqueness, convergence, stability, and well-posedness, each of which requires further assumptions.

Mathematical Preliminaries: This section introduces the mathematical background which is utilized throughout this article, such as b-metric and extended b-metric spaces (8,9,11), the Pompeiu–Hausdorff metric for multi-valued best replies (12), and the construction of the product which gives a complete b-metric to the set of strategies of a finite game.

Metric and Extended b-Metric Spaces: Let X be a nonempty set and $s \geq 1$. A function $d: X \times X \rightarrow [0, \infty)$ is a b-metric if, for all $x, y, z \in X$, (i) $d(x, y) = 0 \Leftrightarrow x = y$; (ii) $d(x, y) = d(y, x)$; and (iii) $d(x, z) \leq s[d(x, y) + d(y, z)]$. For $s = 1$, this becomes the ordinary triangle inequality. The b-metric framework is due to Bakhtin and Czerwik (8,9).

More generally, if $\theta: X \times X \rightarrow [1, \infty)$, a function d_θ satisfying the same definiteness and symmetry conditions and

$$d_\theta(x, z) \leq \theta(x, z)[d_\theta(x, y) + d_\theta(y, z)]$$

is an extended b-metric (11). Taking $\theta \equiv s$ recovers the ordinary b-metric. A sequence $\{x_n\}$ converges to x if $d(x_n, x) \rightarrow 0$, is Cauchy if $d(x_n, x_m) \rightarrow 0$ as $n, m \rightarrow \infty$, and (X, d) is complete provided that all Cauchy sequences converge in X . As a general b-metric is not necessarily joint continuous, any such condition of lower-semicontinuity needed to support the limit process is specified below.

Multi-Valued Mappings and the Pompeiu–Hausdorff Distance

Let $CB(X)$ denote the nonempty closed bounded subsets of X . For $x \in X$ and $A \in CB(X)$,

$$d(x, A) = \inf_{a \in A} d(x, a),$$

and for $A, B \in CB(X)$, the Pompeiu–Hausdorff distance is

$$H_d(A, B) = \max \left\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A) \right\}.$$

A correspondence $F: X \Rightarrow CB(X)$ is a Hausdorff contraction if

$$H_d(F(x), F(y)) \leq \lambda d(x, y)$$

for some $\lambda \in [0, 1)$ and for all $x, y \in X$. The latter notion, proposed classically by Nadler (12) and generalized to b-metric and quasi-contractive frameworks in (13), estimates changes in sets of admissible responses and allows for multi-valued best responses.

Strategy Profiles and the Best-Response Fixed-Point Characterization

Let's consider the non-cooperative game.

$$\Gamma = (I, \{X_i\}_{i \in I}, \{u_i\}_{i \in I}),$$

where $I = \{1, \dots, N\}$, X_i is player i 's strategy set, $u_i: X \rightarrow \mathbb{R}$ is the payoff function, and $X = \prod_i X_i$ is the strategy-profile space. For $x = (x_i, x_{-i}) \in X$,

$$BR_i(x_{-i}) = \arg \max_{y_i \in X_i} u_i(y_i, x_{-i}),$$

and the best response correspondence is

$$B(x) = \prod_i BR_i(x_{-i}).$$

Proposition 2.1. A profile $x^* \in X$ is a Nash equilibrium of Γ if and only if $x^* \in B(x^*)$.

Proof. If x^* is a Nash equilibrium, then $x_i^* \in BR_i(x_{-i}^*)$ for every i , and hence $x^* \in B(x^*)$. Conversely, if $x^* \in B(x^*)$, then x_i^* maximizes player i 's payoff against x_{-i}^* for every i ; therefore, no player can gain through unilateral deviation, so x^* is a Nash equilibrium.

Hence, a fixed point of B is a Nash equilibrium. The separation of the construction and characterization of B from the fixed-point argument ensures that the game theory and geometry assumptions are clear and checkable.

The Product b-Metric on the Strategy-Profile Space

If each X_i carries a b-metric d_i with coefficient $s_i \geq 1$, define on $X = \prod_i X_i$

$$d(x, y) = \max_{i \in I} d_i(x_i, y_i), \quad s = \max_{i \in I} s_i.$$

Then d is a b-metric with coefficient s , since

$$d(x, z) \leq s[d(x, y) + d(y, z)].$$

If every (X_i, d_i) is complete, then (X, d) is also complete: A Cauchy sequence in X is componentwise Cauchy, with each component converging by completeness, and the finite nature of I leading to convergence in the product metric. This technique ensures that the assumptions for the core theorems can be verified using the normal scalar b-metric space. In the cases where the extended b-metric spaces can be scalarized to this form, the results apply to the generalized strategy spaces.

Multi-Valued Contraction Theorem and Nash Equilibrium Existence: The multi-valued contraction theorem on complete b-metric spaces and its application to the collective best-response correspondence are given in this section, leading to Nash equilibrium existence.

Approximate Hausdorff Selection

Lemma 3.1. Let (X, d) be a b-metric space and $A, B \in CB(X)$. If $a \in A$ and $q > 1$, then there exists $b \in B$ such that

$$d(a, b) \leq qH_d(A, B).$$

Proof. Since $d(a, B) \leq H_d(A, B)$ and $d(a, B) = \inf_{b \in B} d(a, b)$, for every $q > 1$ there exists $b \in B$ such that

$$d(a, b) < q d(a, B) \leq qH_d(A, B).$$

The factor $q > 1$ accounts for the possibility that the infimum defining $d(a, B)$ is not attained and enables the construction of the iterative sequence used below.

A Multi-Valued Fixed-Point Theorem in a Complete b-Metric Space: Theorem 3.1 (Multi-valued b-metric contraction theorem; cf. (12,13)). Let (X, d) be a complete b-metric space with coefficient $s \geq 1$, and let $T: X \Rightarrow CB(X)$ satisfy

$$H_d(T(x)T(y)) \leq \lambda d(x, y), \lambda \in [0, 1),$$

for all $x, y \in X$, with
 $s\sqrt{\lambda} < 1$.

Then T has a fixed point; that is, there exists $x^* \in X$ such that $x^* \in T(x^*)$.

Proof. Choose $x_0 \in X$ and $x_1 \in T(x_0)$. Given $x_n \in T(x_{n-1})$, Lemma 3.1 yields $x_{n+1} \in T(x_n)$ such that

$$d(x_n, x_{n+1}) \leq q\lambda d(x_{n-1}, x_n).$$

Choose $q > 1$ sufficiently close to 1 and set $\rho = q\lambda$ so that $s\sqrt{\rho} < 1$. Then

$$d(x_n, x_{n+1}) \leq \rho^n d(x_0, x_1).$$

For $m > n$, repeated application of the b-metric inequality gives

$$d(x_n, x_m) \leq \frac{s\rho^n}{1 - s\rho} d(x_0, x_1) \rightarrow 0.$$

Hence $\{x_n\}$ is Cauchy and, by completeness, $x_n \rightarrow x^* \in X$. Since $x_{n+1} \in T(x_n)$,

$$d(x_{n+1}, T(x^*)) \leq H_d(T(x_n)T(x^*)) \leq \lambda d(x_n, x^*) \rightarrow 0.$$

Under the stated lower-semicontinuity condition for the distance-to-set map,

$$d(x^*, T(x^*)) \leq \liminf_n d(x_{n+1}, T(x^*)) = 0.$$

As $T(x^*)$ is closed, $x^* \in T(x^*)$.

The condition $s\sqrt{\lambda} < 1$ expresses the balance between contraction and the amplification introduced by the relaxed triangle inequality. It reduces to $\lambda < 1$ when $s = 1$, while for $s > 1$, the contraction must additionally compensate for the b-metric coefficient.

Nash Equilibrium Existence Theorem: Theorem 3.1 applied to the aggregate best response correspondence B yields the critical equilibrium theorem.

Theorem 3.2 (Nash equilibrium under b-metric contractivity). Let Γ be a finite non-cooperative game with strategy-profile space $X = \prod_i X_i$, and suppose:

- (A1) (X, d) is a complete b-metric space with coefficient $s \geq 1$, constructed as in Section 2.4;
- (A2) $B(x) \in CB(X)$ for every $x \in X$;
- (A3) $H_d(B(x)B(y)) \leq \lambda d(x, y)$ for some $\lambda \in [0, 1)$ and all $x, y \in X$;
- (A4) $s\sqrt{\lambda} < 1$;
- (A5) the distance-to-set operation for $B(x^*)$ is lower semicontinuous along convergent sequences.

Then Γ possesses at least one Nash equilibrium.

Proof. By (A1)–(A4), Theorem 3.1 applies to B and yields $x^* \in B(x^*)$; (A5) validates the limiting step. By Proposition 2.1, x^* is a Nash equilibrium of Γ . Contraction implies that the strategy profile in close proximity creates the set of simultaneous best responses in the neighborhood. The threshold $s\sqrt{\lambda} < 1$ and thus ensures the dominance of the contraction in the metric magnification. This theorem proves existence, but does not prove uniqueness, convergence, or stability, which need further conditions discussed in Sections 4 and 5.

Uniqueness, Comparison, and Scope: Theorem 3.2 is a statement of existence alone. The current section explores the further implications arising under stronger conditions, contrasts the theory with the classical equilibrium existence theories, and explains the conditions under which the theory applies.

Uniqueness under a Global Contraction

Theorem 4.1. Under the hypotheses of Theorem 3.2, if the constant λ in (A3) applies globally as a Hausdorff contraction of B on X , then the Nash equilibrium is unique.

Proof. Suppose x^*, y^* are Nash equilibria, so $x^* \in B(x^*)$ and $y^* \in B(y^*)$. Then

$$d(x^*, y^*) \leq H_d(B(x^*), B(y^*)) \leq \lambda d(x^*, y^*).$$

Thus, $(1 - \lambda)d(x^*, y^*) \leq 0$. Since $\lambda < 1$, $d(x^*, y^*) = 0$, and hence $x^* = y^*$.

Consequently, global Hausdorff contraction gives us a stronger result than existence alone. Whereas completeness and closedness can give several fixed points, the global contraction gives us uniqueness.

Comparison with Classical Equilibrium Existence Results: However, the suggested framework is not intended to replace but supplement classical topological methods. Table 1 gives a brief comparison of the major structural differences between classical and metric-contractive equilibrium existence frameworks.

Table 1. Comparison of Classical and Metric-Contractive Equilibrium Existence Frameworks

Framework	Principal structure	Main mechanism	Typical requirement
Brouwer [1]	Euclidean / topological vector space	Single-valued fixed point	Continuity, compactness, convexity
Kakutani [2]	Topological vector space	Multi-valued fixed point	Compact convex values, u.s.c.
Fan–Glicksberg [5,6]	Locally convex space	Generalized topological fixed point	Compactness, convexity, u.s.c.
Nadler [12]	Complete metric space	Multi-valued Hausdorff contraction	Contraction constant $\lambda < 1$
Present framework [Thm. 3.2]	Complete b-metric space	Multi-valued Hausdorff contraction	Completeness, closed bounded values, $s\sqrt{\lambda} < 1$

Sources: [1,2,5,6,12,13]; present framework as derived in Section 3.

As seen from Table 1, the classical approach takes advantage mostly of convexity, compactness, and continuity while Theorem 3.2 uses metric completeness and contractivity. Neither approach is better than the other; the choice of approach depends on whether the game setup fits topological or contractive metric considerations.

Scope and Limitations: Theorem 3.2 guarantees existence only under (A1)–(A5) and does not imply that every generalized b-metric game has a Nash equilibrium. In particular, nonemptiness of $BR_i(x_{-i})$ is not automatic and may require game-theoretic compactness and continuity conditions, such as Weierstrass' theorem when $F_i(x_{-i})$ is compact and u_i is continuous in the player's own strategy. The closedness and boundedness of $B(x)$ must hold for every x , and the threshold $s\sqrt{\lambda} < 1$ must be verified rather than merely $\lambda < 1$. Each semicontinuity assumption that is made in the limiting proof has to be verified separately since a general b-metric is not necessarily jointly continuous. Uniqueness needs the proof via Theorem 4.1, whereas convergence of a best-response algorithm is an entirely different question covered in Section 5.

Constructive Approximation, Stability, and Well-Posedness: In addition to existence, the contraction criterion of Theorem 3.2 has constructive consequences: best-response iterations converge geometrically to an equilibrium; close best-response correspondences imply close equilibria; and, in case of uniqueness, approximate equilibria converge to the equilibrium. Theorems employ the scalar b-metric of Section 2.4.

Iterative Selection and Geometric Convergence: Given $x_0 \in X$, choose $x_1 \in B(x_0)$, and for $n \geq 1$, use Lemma 3.1 to choose $x_{n+1} \in B(x_n)$ such that

$$d(x_n, x_{n+1}) \leq q H_d(B(x_{n-1}), B(x_n)),$$

where $q > 1$. The Hausdorff contraction then gives

$$d(x_n, x_{n+1}) \leq q\lambda d(x_{n-1}, x_n).$$

Theorem 5.1 (Quantitative convergence). Under the hypotheses of Theorem 3.2, the sequence $\{x_n\}$ converges to a fixed point x^* of B , hence to a Nash equilibrium, and

$$d(x_n, x^*) \leq \frac{s\rho^n}{1 - s\rho} d(x_0, x_1), \quad \rho = q\lambda, \quad s\rho < 1.$$

Proof (sketch). The estimate follows from the Cauchy bound in Theorem 3.1 with $T = B$. Letting $m \rightarrow \infty$ and using $x_{n+1} \in B(x_n)$ together with the lower-semicontinuity condition (A5) gives $x^* \in B(x^*)$. The bound provides a geometric or R-linear convergence rate determined by both ρ and s , instead of solely depending on λ . Approximate solutions that satisfy

$$d(x_n, x_{n+1}) \leq \rho d(x_{n-1}, x_n) + \varepsilon_n,$$

with $\rho < 1/\square$ and $\sum \varepsilon_n < \infty$, The error accumulation is still governed by the same geometric process, hence exhibiting Cauchy behavior with respect to the corresponding b-metric convergence.

Perturbation Stability of the Equilibrium: Let $B_1, B_2: X \Rightarrow CB(X)$ satisfy Hausdorff contractions with the same $\lambda \in [0, 1)$, assume $s^2\lambda < 1$, and suppose

$$H_d(B_1(x), B_2(x)) \leq \varepsilon$$

for all $x \in X$.

Theorem 5.2 (Perturbation stability). If x_1^* and x_2^* are fixed points of B_1 and B_2 , respectively, then

$$d(x_1^*, x_2^*) \leq \frac{s^2}{1 - s^2\lambda} \varepsilon.$$

Proof. Since $x_2^* \in B_2(x_2^*)$,

$$d(x_1^*, x_2^*) \leq sH_d(B_1(x_1^*), B_2(x_2^*)).$$

Using the b-metric inequality and the assumptions,

$$H_d(B_1(x_1^*), B_2(x_2^*)) \leq s[\varepsilon + \lambda d(x_1^*, x_2^*)].$$

Hence

$$(1 - s^2\lambda)d(x_1^*, x_2^*) \leq s^2\varepsilon,$$

which gives the stated bound when $s^2\lambda < 1$.

The factor s^2 is retained because the proof applies the relaxed triangle inequality twice; therefore, the conservative bound is $s^2\varepsilon/(1 - s^2\lambda)$.

Tykhonov-Type Well-Posedness: The equilibrium problem for B is Tykhonov well-posed if it has a unique Nash equilibrium x^* and every sequence $\{y_n\}$ satisfying

$$d(y_n, B(y_n)) \rightarrow 0$$

also satisfies $y_n \rightarrow x^*$.

Theorem 5.3. Under the hypotheses of Theorem 3.2 with $s\lambda < 1$, then the Nash Equilibrium Problem is Tykhonov well-posed if the equilibrium is unique and the distance function is consistent with the limiting argument.

Proof (sketch). For an approximate-fixed-point sequence, choose $z_n \in B(y_n)$ with $d(y_n, z_n) \rightarrow 0$. Since $x^* \in B(x^*)$, Hausdorff contractivity gives

$$d(z_n, B(x^*)) \leq \lambda d(y_n, x^*).$$

Using Lemma 3.1 to select points in $B(x^*)$ and applying the b-metric inequality yields

$$d(y_n, x^*) \leq C[d(y_n, B(y_n)) + \varepsilon_n] + C'\lambda d(y_n, x^*),$$

where C, C' depend only on s and the selection convention. Under $s\lambda < 1$, the final term is absorbed on the left, giving $d(y_n, x^*) \rightarrow 0$. Therefore, well-posedness implies both the existence and uniqueness conditions, together with the control of approximate fixed points using the contraction property.

As such, from Sections 3 to 5, there is a hierarchy: the existence condition of Theorem 3.2 does not imply the uniqueness condition (Theorem 4.1), the convergence rate condition (Theorem 5.1), the stability condition (Theorem 5.2), and the well-posedness condition (Theorem 5.3).

Numerical and Visual Illustration: The following analysis demonstrates the working of the contraction principle through an example involving two players and analyzes the dependence of the threshold value $s\sqrt{\lambda} < 1$. The numbers and figures presented here are meant to complement the analytical approach and not substitute the proof in Section 3.

Worked Example and Comparative Classification

Let $X_1 = X_2 = [0, 1]$, $X = X_1 \times X_2$, with product b-metric

$$d(x, y) = \max \{d_1(x_1, y_1), d_2(x_2, y_2)\}$$

and common coefficient $s=1.2$ for the component spaces (8,9). Assume that the aggregate best-response correspondence satisfies

$$H_d(B(x), B(y)) \leq \lambda d(x, y), \lambda = 0.50.$$

Then

$$s\sqrt{\lambda} = 1.2\sqrt{0.50} \approx 0.8485 < 1,$$

so that the threshold of Theorem 3.2 is met. Table 2 provides the setting of this construction within the context of classical and generalized metrics/topologies.

As shown in Table 2 below, the current model is different from classical metrics and topology because the b-metric coefficient is included within the definition of the contraction. The condition $s\sqrt{\lambda} < 1$ and thus represents the crucial factor in equilibrium analysis under the suggested metric space formulation.

Table 2. Comparative Classification of Metric and Topological Equilibrium Structures

Structure	Contraction mechanism	Typical equilibrium interpretation
Classical metric space	Hausdorff contraction, $H_d \leq \lambda d, \lambda < 1$ [12]	Nash equilibrium via fixed point of best response
Complete b-metric space [8,9]	Hausdorff contraction with threshold $s\sqrt{\lambda} < 1$ [12,13]	Nash equilibrium not necessarily governed by classical geometry
Extended b-metric space [11]	Contractions incorporating variable control $\theta(x,z)$	Equilibrium under more general distance structures
G-convex / abstract-convex space	KKM- or selection-type conditions	Equilibrium in abstract economies (not required by Thm. 3.2)

Iterative ConvergenceL: Taking $x_0 = (0, 0)$ and the admissible selection

$$x_n = (0.5, (1 - 2^{-n})0.5(1 - 2^{-n})),$$

the sequence converges to $x^* = (0.5, 0.5)$, with

$$d(x_n, x^*) = 0.5(0.5)^n.$$

This proves the geometric decay explicitly stated in Theorem 5.1 (Section 5.1). The iterative process and distance from the equilibrium profile are provided in Table 3.

Table 3. Iterative Best-Response Trajectory and Distance to the Equilibrium Profile

Iteration n	x_n	$d(x_n, x^*)$
0	(0.0000, 0.0000)	0.500000
1	(0.2500, 0.2500)	0.250000
2	(0.3750, 0.3750)	0.125000
3	(0.4375, 0.4375)	0.062500
4	(0.46875, 0.46875)	0.031250
5	(0.484375, 0.484375)	0.015625
10	(0.499512, 0.499512)	0.000488

Illustrative trajectory constructed under the contraction condition of Theorem 5.1 (12). The values in Table 3 demonstrate a constant decrease in terms of a geometric series for the distance to x^* , with the error halved at every step. Such numerical performance is in line with the theory of the method developed in Section 5.1.

Sensitivity of the Convergence Condition

For fixed $\lambda = 0.50$, Table 4 reports

$$R(s) = s\sqrt{\lambda}$$

for various values of s. This is true only if

$$s < \frac{1}{\sqrt{\lambda}} = \sqrt{2} \approx 1.4142.$$

Therefore, raising the b-metric coefficient will narrow down the permissible contraction values. This shows how the geometry moves away from being just a metric (8,9). Numerical results are shown in Table 4. Threshold computed from Theorem 3.1, condition $s\sqrt{\lambda} < 1$ (12,13). As demonstrated in Table 4, the threshold remains satisfied through $s = 1.4$ but fails at $s = 1.5$. This shows that a greater b-metric coefficient requires a greater contraction for the necessary condition.

Table 4. Sensitivity of $R(s) = s\sqrt{\lambda}$ to the b-Metric Coefficient ($\lambda = 0.50$)

s	$R(s) = s\sqrt{0.50}$	Sufficient condition $s\sqrt{\lambda} < 1$
1.0	0.7071	Satisfied
1.1	0.7778	Satisfied
1.2	0.8485	Satisfied
1.3	0.9192	Satisfied
1.4	0.9899	Satisfied
1.5	1.0607	Not satisfied

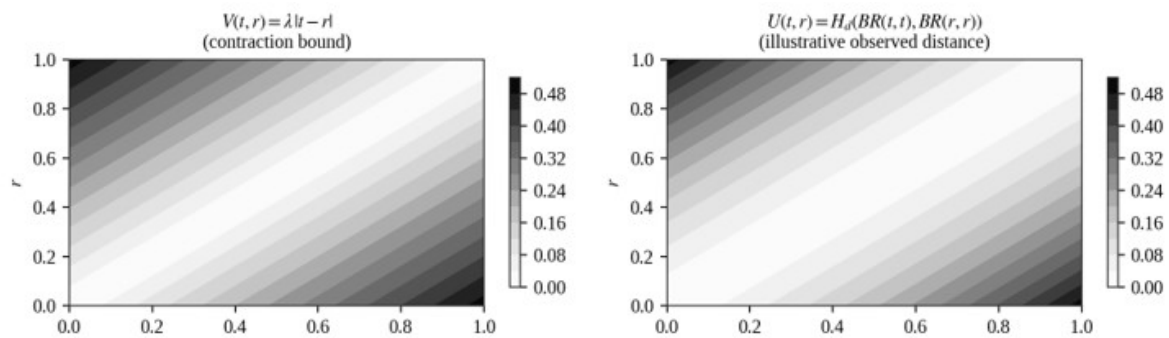
Visual Diagnostics: The numerical results are further illustrated graphically in Figure 1 and Figure 2. Figure 1 considers the diagonal restriction $x = (t \ t)$, $y = (r \ r)$ and compares the contraction bound

$$V(t \ r) = \lambda |t - r|$$

alongside the Hausdorff distance example

$$U(t \ r) = H_d(B(t \ t)B(r \ r)).$$

This test checks whether the observed distance is contained within the presumed contraction. Based on the contraction condition of Theorem 3.1 and the illustration of Section 6.1 (12,13).

**Figure 1. Best-Response Distance Diagnostic over $[0, 1]^2$ ($\lambda = 0.50$)**

From Figure 1, it is clear that the proposed diagnostic satisfies $U \leq V$ pointwise in the selected diagonal section, in accordance with the given Hausdorff contraction condition.

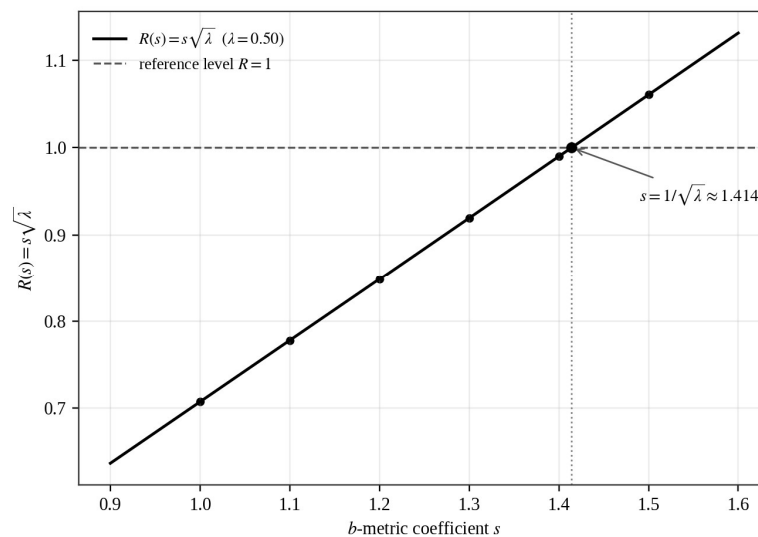
**Figure 2. Sensitivity of $R(s) = s\sqrt{\lambda}$ to the b-Metric Coefficient s**

Figure 2 gives the alternate sensitivity visualization by plotting

$$R(s) = s\sqrt{\lambda}$$

against s , along with the reference value of $R=1$. It determines the crucial coefficient

$$s = \frac{1}{\sqrt{\lambda}} \approx 1.4142,$$

that is equivalent to the threshold value given in Table 4. Threshold $s\sqrt{\lambda} < 1$ as derived in Theorem 3.1 (8,9,12). Taken together, Figures 1 and 2 offer graphic evidence regarding the two main numerical features of the framework: the validity of the assumed contraction function and the dependence of the sufficient convergence condition on the b-metric parameter. No uniqueness is implied by this numerical example. For a true multi-valued best response function, neither Hausdorff contractivity nor the convergence of one particular trajectory can be taken to imply uniqueness of the fixed point (as seen in Theorem 4.1) or global stability, respectively. This is because the additional assumptions made in Sections 4-5 are needed to achieve both of these features.

CONCLUSION AND FUTURE RESEARCH DIRECTIONS

In this work, we establish an existence theorem for Nash equilibrium based on fixed-point theory, where the strategy set is a complete b-metric space and the collective best-response map is a closed, bounded-valued multi-valued mapping. Under the Pompeiu–Hausdorff contractivity condition and threshold $s\sqrt{\lambda} < 1$, There is a fixed point in the correspondence that is identical to the Nash equilibrium (Theorem 3.2). More assumptions produce additional conclusions: global contraction implies uniqueness (Theorem 4.1), iterative best-response selections converge geometrically at a rate determined by λ and s (Theorem 5.1), Perturbations of the best response correspondence generate controlled perturbations of the equilibria (Theorem 5.2), while uniqueness along with contraction guarantees Tykhonov well-posedness. The condition $s\sqrt{\lambda} < 1$ therefore quantifies the balance between contraction strength and the amplification caused by the relaxed triangle inequality, reducing to the classical condition $\lambda < 1$ when $s = 1$. The contribution is purely methodological and does not assert the superiority of the generalized b-metric approach over classical equilibrium theory in any way. In cases where compactness, convexity, and upper semi-continuity are available, the approaches of Brouwer, Kakutani, and Fan-Glicksberg (1,2,5,6) are perfectly suitable, while the current one is more appropriate for environments where metric completeness and contraction of best response behavior constitute better assumptions. It also incorporates sensitivity of best responses into the sufficient condition for existence, in line with the connection between fixed point and equilibrium theorems (14,15). The main drawbacks include the use of a particular scalar case of b-metric instead of a general variable-control b-metric theorem (11), the restriction of convergence results to admissible best response procedures rather than general computational algorithms, and the example in Section 6 being illustrative rather than computational.

The future research can consider more stringent fixed-point conditions in the extended/variable control b-metric spaces (11), find other conditions such as single-valuedness, strict payoffs, and selection continuity ensuring uniqueness, and generalize the results to approximate best responses and ε -Nash equilibria. Other research paths include probabilistic generalized metric games, best response iterations in dynamic and distributed environment, and extension of generalized Nash equilibrium problems with constraints. The latter case will involve certain assumptions and fixed point argument rather than generalization of results stated here. Overall, this study demonstrates that the use of generalized b-metric spaces as a quantitative framework for Nash equilibrium through multivalued fixed-point methods is feasible. The condition $s\sqrt{\lambda} < 1$ serves as the central verifiable criterion linking the geometry of the strategy space with the contractive behaviour of best responses. This provides an exact metric approach from best response correspondence to the existence of equilibrium and forms a basis for exploration of broader distance systems and approximation and dynamic methods for equilibrium. This research thus contributes a mathematically sound framework for making connections between multi-valued best response maps and Nash equilibrium using the generalized b-metric fixed point theory. By explicitly relating the contraction parameter to the b-

metric coefficient, the condition $s\sqrt{\lambda} < 1$ provides a clear condition for the existence of an equilibrium point and allows the further quantification of issues such as convergence, stability, and well-posedness. The model therefore adds a new metric perspective to the study of equilibrium and a foundation for extending fixed-point techniques to more general games and metrics. In summary, the findings and analysis show that fixed-point approaches in generalized b-metric spaces can be used as a unified approach for analyzing Nash equilibria. The proposed condition $s\sqrt{\lambda} < 1$ provides an easily checked test that can be used in determining the existence of equilibrium as well as making quantitative predictions concerning convergence and stability under suitable conditions. The research thus broadens the use of metric fixed point theory in strategic games.

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