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RESEARCH ARTICLE

ON THE QUINTIC EQUATION WITH FIVE UNKNOWNS $\left[x^3 - y^3 = z^3 - w^3 + 6t^5\right]$

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ABSTRACT

We obtain infinitely many non-zero integer quintuples (x, y, z, w, t) satisfying the Quintic Equation with five unknowns $x^3 - y^3 = z^3 - w^3 + 6t^5$. Various interesting properties between the values of x, y, z, w, t and special polygonal and pyramidal numbers are presented.

INTRODUCTION

The theory of Diophantine equations offers a rich variety of fascinating problems. In particular, Quintic equations, homogeneous and non-homogeneous have aroused the interest of numerous mathematicians since antiquity [1-3]. For illustration, one may refer [4-6] for Quintic equation with three unknowns, [7] for Quintic with four unknowns and [8-10] for Quintic equation with five unknowns. This paper concerns with the problem of determining non-trivial integral solutions of the non-homogeneous Quintic equation with five unknowns given by $x^3 - y^3 = z^3 - w^3 + 6t^5$. A few relations among the solutions are presented.

METHOD OF ANALYS

The Quintic Diophantine Equation with five unknowns to be solved for its non zero distinct integral solutions is

$$x^3 - y^3 = z^3 - w^3 + 6t^5 \quad (1)$$

Different patterns of solutions of (1) are presented below.

Pattern I:

Introduction of the transformations

$$x = c+1, y = c-1$$
$$z = a+1, w = a-1 \quad (2)$$

in (1) leads to $c^2 = a^2 + t^5$

Case: i

Let $(c+a) = A^4 \quad (c-a) = A$

Hence, the corresponding solutions of (1) are

$$x(A) = \frac{1}{2}(A^4 + A + 2)$$
$$y(A) = \frac{1}{2}(A^4 + A - 2)$$
$$z(A) = \frac{1}{2}(A^4 - A + 2)$$
$$w(A) = \frac{1}{2}(A^4 - A - 2)$$
$$t(A) = A$$

x,y,z,w and t are integers, for all values of A.

Properties:

- $x(A) - y(A) + z(A) - w(A) \equiv 0 \pmod 4$
- $2(x(A) - z(A)) = gn_A + 1$
- $y(A(A+1)) - w(A(A+1)) = x(A(A+1)) - z(A(A+1)) = \Pr_A$

Each of the following expressions represents a Nasty number.

a) $3\{x(A) + y(A) + z(A) + w(A)\}$
b) $6\{y(A) + w(A) + 2\}$
c) $6\{x(A) + z(A) - 2\}$
d) $6\{x(A) + w(A)\}$
e) $6\{y(A) + z(A)\}$

- $z(A) + w(A) + t(A)$ is a Biquadratic integer.

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Case: ii

Take $(c+a)=A^3 \quad (c-a)=A^2$

Hence, the corresponding integer solutions of (1) are

$$x(A)=\frac{1}{2}(A^3+A^2+2)$$

$$y(A)=\frac{1}{2}(A^3+A^2-2)$$

$$z(A)=\frac{1}{2}(A^3-A^2+2)$$

$$w(A)=\frac{1}{2}(A^3-A^2-2)$$

$$t(A)=A$$

Properties:

- $x(A)+y(A)=2P_A^5$
- $x(A)+y(A)+z(A)+w(A)-t(A)=SO_A$
- $x(A)=ct_{A,A}$

Each of the following expressions represents a Nasty number.

a) $6\{y(A)-w(A)\}$
b) $6\{x(A)-z(A)\}$
c) $3\{x(A)+y(A)-z(A)-w(A)\}$

Each of the following expressions represents a Cubical integer.

- $y(A)+z(A)$
- $x(A)+w(A)$

Case: iii

Let $(c+a)=A^5 \quad (c-a)=1$

Hence, the corresponding solutions of (1) are

$$x(A)=\frac{1}{2}(A^5+3)$$

$$y(A)=\frac{1}{2}(A^5-1)$$

$$z(A)=\frac{1}{2}(A^5+1)$$

$$w(A)=\frac{1}{2}(A^5-3)$$

$$t(A)=A$$

As our aim is on finding integer solutions, it is seen that the values of x,y,z,w and t are integers only when A is odd. ie $A=2k+1$.Thus, the corresponding solutions of (1) are

$$x(k)=16k^5+40k^4+40k^3+20k^2+5k+2$$

$$x(k)=16k^5+40k^4+40k^3+20k^2+5k$$

$$x(k)=16k^5+40k^4+40k^3+20k^2+5k+1$$

$$x(k)=16k^5+40k^4+40k^3+20k^2+5k-1$$

$$t(k)=2k+1$$

Properties:

- $x(A)-y(A)+z(A)+w(A)+t(A)\equiv 4(\mathrm{mod}\, A)$
- $x(A)-y(A)+w(A)-z(A)=0$

Each of the following expressions represents a Nasty number.

a) $6\{x(A)-y(A)\}$
b) $6\{x(A)-y(A)+z(A)-w(A)\}$

- $2\{x(A)-y(A)-z(A)-w(A)$ is a cubical integer.
- $16\{x(A)+y(A)+w(A)+z(A)\}$ is a quintic integer.

PATTERN II:

Introduction of another transformations

$$x=u+v \qquad w=u-v \qquad t=ku$$
$$y=u+p \qquad z=u-p$$

in (1) leads to $v^2=p^2+k^5u^4$ (3)

Case: i

Let $(v+p)=k^5A^4 \quad (v-p)=1$

Hence, the corresponding solutions of (1) are

$$x(A)=\frac{1}{2}(2A+k^5A^4+1)$$

$$y(A)=\frac{1}{2}(2A+k^5A^4-1)$$

$$z(A)=\frac{1}{2}(2A-k^5A^4+1)$$

$$w(A)=\frac{1}{2}(2A-k^5A^4-1)$$

$$t(A)=kA$$

The quintuple (x,y,z,w,t) is an integer, when both A and k are odd.

Case: ii

Let $(v+p)=k^5A^3 \quad (v-p)=A$

Hence, the corresponding solutions of (1) are

$$x(A)=\frac{1}{2}(3A+k^5A^3)$$
$$y(A)=\frac{1}{2}(A+k^5A^3)$$
$$z(A)=\frac{1}{2}(3A-k^5A^3)$$
$$w(A)=\frac{1}{2}(A-k^5A^3)$$
$$t(A)=kA$$

The quintuple (x,y,z,w,t) is an integer, when both A and k are odd.

Case: iii

Consider $(v+p)=k^5A^2 \quad (v-p)=A^2$

Hence, the corresponding solutions of (1) are

$$x(A)=\frac{1}{2}(2A+A^2(k^5+1))$$
$$y(A)=\frac{1}{2}(2A+A^2(k^5-1))$$
$$z(A)=\frac{1}{2}(2A+A^2(1-k^5))$$
$$w(A)=\frac{1}{2}(2A-A^2(k^5+1))$$
$$t(A)=kA$$

The quintuple (x,y,z,w,t) is an integer, when k is odd.

Case: iv

Take $(v+p)=A^4 \quad (v-p)=k^5$

Hence, the corresponding solutions of (1) are

$$x(A)=\frac{1}{2}(2A+A^4+k^5)$$
$$y(A)=\frac{1}{2}(2A+A^4-k^5)$$
$$z(A)=\frac{1}{2}(2A-A^4+k^5)$$
$$w(A)=\frac{1}{2}(2A-A^4-k^5)$$
$$t(A)=kA$$

The values of x,y,z,w and t are integers ,when both A and k are of the same parity.

Case: v

Assume $(v+p)=k^3A^2 \quad (v-p)=k^2A^2$

Thus, the corresponding solutions of (1) are

$$x(A)=\frac{1}{2}(2A+A^2(k^3+k^2))$$
$$y(A)=\frac{1}{2}(2A+A^2(k^3-k^2))$$
$$z(A)=\frac{1}{2}(2A+A^2(k^2-k^3))$$
$$w(A)=\frac{1}{2}(2A-A^2(k^3+k^2))$$
$$t(A)=kA$$

The values of x,y,z,w and t) are integer ,when A is even.

PATTERN III:

When k$\neq$ a perfect square

(3) is of the form $z^2=Dx^2+y^2$

Hence the solutions of (3) is

$$u^2=2rs$$
$$v=k^5r^2+s^2$$
$$p=k^5r^2-s^2$$

Our interest is on finding integer solutions, so take $r=2^{2\alpha-1}s$.

Hence, the corresponding nonzero distinct integral solutions of (1) are given by

$$x=2^{\alpha}s+(k^5 2^{4\alpha-2}+1)s^2$$
$$y=2^{\alpha}s+(k^5 2^{4\alpha-2}-1)s^2$$
$$z=2^{\alpha}s-(k^5 2^{4\alpha-2}-1)s^2$$
$$w=2^{\alpha}s-(k^5 2^{4\alpha-2}+1)s^2$$
$$t=k2^{\alpha}s$$

If $k=\alpha^2$, (3) leads to $v^2=p^2+(\alpha^5u^2)^2$ (4)

which is satisfied by

$$\alpha^5u^2=2rs$$
$$v=r^2+s^2\ ,\ r>s>0 \qquad (5)$$
$$p=r^2-s^2$$

Let as assume that $r=2^{2\beta-1}\alpha^5R^2s$

Then (5) becomes

$$u=2^{\beta}Rs$$
$$p=(2^{4\beta-2}\alpha^{10}R^4-1)s^2$$
$$v=(2^{4\beta-2}\alpha^{10}R^4+1)s^2$$

Hence the corresponding solutions of (1) is

$$x = 2^{\beta} Rs + (2^{4\beta-2}\alpha^{10}R^4 + 1)s^2$$
$$y = 2^{\beta} Rs + (2^{4\beta-2}\alpha^{10}R^4 - 1)s^2$$
$$z = 2^{\beta} Rs - (2^{4\beta-2}\alpha^{10}R^4 - 1)s^2$$
$$w = 2^{\beta} Rs - (2^{4\beta-2}\alpha^{10}R^4 + 1)s^2$$
$$t = \alpha^2 2^{\beta} Rs$$

Also, the solutions of (4) are

$$\alpha^5 u^2 = r^2 - s^2$$
$$v = r^2 + s^2$$
$$p = 2rs$$

Let $r = \alpha^5 R, s = \alpha^5 S$

Then (6) becomes $u^2 = \alpha^5 (R^2 - S^2)$ (7)

Again taking $R = \alpha^3 \overline{R}, S = \alpha^3 \overline{S}$ in (7), it leads to

$$u^2 = \alpha^{14}(\overline{R}^2 - \overline{S}^2)$$

Consider $\overline{R} = M^2 + N^2, \overline{S} = 2MN$

Then

$$u = \alpha^7 (M^2 - N^2)$$
$$v = \alpha^{34}(M^4 + N^4 + 6N^2 + M^2)$$
$$p = 4\alpha^{31} MN(M^2 + N^2)$$

Thus the corresponding solutions of (1) are

$$x = \alpha^7 (M^2 - N^2) + \alpha^{34}(M^4 + N^4 + 6N^2 + M^2)$$
$$y = \alpha^7 (M^2 - N^2) + 4\alpha^{31} MN(M^2 + N^2)$$
$$z = \alpha^7 (M^2 - N^2) - 4\alpha^{31} MN(M^2 + N^2)$$
$$w = \alpha^7 (M^2 - N^2) - \alpha^{34}(M^4 + N^4 + 6N^2 + M^2)$$
$$t = \alpha^9 (M^2 - N^2)$$

REMARKABLE OBSERVATIONS

Employing the solutions (x, y, z, w, t) of (1), a few observations among the special polygonal and pyramidal numbers are exhibited below

1. $$\left[\frac{3P^3_{x-2}}{t_{3,x-2}}\right]^3 - \left[\frac{P^5_y}{t_{3,y}}\right]^3 + \left[\frac{6P^4_{w-1}}{t_{3,2w-2}}\right]^3 - 3\left[\frac{12p^5_z}{s_{z-1}-1}\right]^3 4 \equiv 0 \pmod 6$$

2. $$\left[\frac{t_{3,2x-1}}{gn_x}\right]^3 - \left[\frac{3(p^4_{y-1} - p^3_{y-1})}{t_{3,y-2}}\right]^3 + \left[\frac{36p^3_{w-2}}{s_{w-1}-1}\right]^3 - 6\left[\frac{4P^5_t}{t_{3,t}}\right]^5$$

is a cubical integer.

3. $$36\left[\frac{p^3_{w-2}}{s_{w-1}-1}\right]^3 - 6^2\left[\frac{P^3_{z-2}}{t_{3,z-2}}\right]^3 + 36\left[\frac{P^4_x}{t_{6,x+1}}\right]^3 - 6^2\left[\frac{P^4_{y-1}}{t_{3,2y-2}}\right]^3$$

is a quintic integer

Conclusion

To conclude, one may search for other patterns of solutions and their corresponding properties.

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